M2R PSA **03 février 2010, durée 3h00**
Physics beyond the standard model
All documents allowed

PART I: Supersymmetry

Hints:

- $\eta^{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$ is the (flat) space-time metric.
- $\sigma^\mu = (I_2, \vec{\sigma})$, $\bar{\sigma}^\nu = (I_2, -\vec{\sigma})$ with $I_2 = \mathrm{diag}[1, 1]$ and the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma^2 = \begin{pmatrix} 0 & \text{-i} \\ \text{i} & 0 \end{pmatrix} , \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix}$$

- $\sigma^i = \sigma^{i\dagger}$ (Hermitian)
- $\mathrm{Tr}\, \sigma^i = 0$ (Traceless)
- $\{\sigma^i, \sigma^j\} = 2\ \delta^{ij}\ I_2$ (Clifford algebra)
- $[\sigma^i, \sigma^j] = 2\ i\epsilon^{ijk}\sigma^k$ (Lie algebra)
- $\sigma^i\sigma^j = \frac{1}{2}\ \{\sigma^i, \sigma^j\} + \frac{1}{2}\ [\sigma^i, \sigma^j] = \delta^{ij} I_2 + i\epsilon^{ijk}\sigma^k$

Problem 1: The supersymmetric ground state

a) Show that $\mathrm{Tr}(\sigma^\mu \bar\sigma^\nu) \equiv \sigma^\mu_{\alpha\dot\beta}\, \bar\sigma^{\nu\dot\beta\alpha} = 2\,\eta^{\mu\nu}$.

b) Show that $\{Q_\alpha, \bar Q_{\dot\beta}\}\, \bar\sigma^{\nu\dot\beta\alpha} = 4P^\nu$.

c) Show that the operator $H := P^0$ has real and non-negative eigenvalues $E \geq 0$.

d) If $|0\rangle$ is the ground state (vacuum state) show that

$$\langle 0|H|0\rangle = 0 \quad \Leftrightarrow \quad Q_\alpha\, |0\rangle = \bar Q_{\dot\alpha}\, |0\rangle = 0 \quad (\alpha, \dot\alpha = 1, 2)\,.$$

Conclusion: A ground state with positive energy breaks supersymmetry spontaneously: $[H, Q_\alpha] = 0$ (SUSY algebra) but $Q_\alpha\, |0\rangle \neq 0$.

Problem 2: Number of bosonic and fermionic degrees of freedom in SUSY multiplets

Recall the Casimir operators of the Poincaré algebra, P^2 and W^2 where W_μ is the Pauli-Lubanski vector. Note that: $[P^2, Q_\alpha] = [P^2, \bar Q_{\dot\alpha}] = 0$ but $[W^2, Q_\alpha] \neq 0$, $[W^2, \bar Q_{\dot\alpha}] \neq 0$. Thus, irreducible (and therefore also reducible) representations of the supersymmetry algebra will contain states with different spins. Schematically, we can write

$$Q_\alpha\, |B\rangle = |F\rangle \;,\; Q_\alpha\, |F\rangle = |B\rangle \;,$$

where $|B\rangle$ is a bosonic and $|F\rangle$ a fermionic state.

Definition: $(-1)^{N_F}$ is an operator defined such that

$$(-1)^{N_F}\, |B\rangle = +\, |B\rangle \;,\; (-1)^{N_F}\, |F\rangle = -\, |F\rangle \;.$$

a) Show that $Q_\alpha(-1)^{N_F} = -(-1)^{N_F} Q_\alpha$.

b) Show that $\mathrm{Tr}[(-1)^{N_F}] = 0$ (for fixed non-zero P_μ) where the trace takes all states of the representation/multiplet into account. (Hint: Evaluate $\mathrm{Tr}[(-1)^{N_F}\{Q_\alpha, \bar Q_{\dot\beta}\}]$ directly and by using the right side of the corresponding supersymmetry algebra relation.)

c) Conclude that every representation of the supersymmetry algebra contains an equal number of bosonic and fermionic states.