M2R PSA
Physics beyond the standard model
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PART I: Supersymmetry

Formulae:

Space-time metric:

- $\eta^{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$

In the Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\alpha} \\ (\bar\sigma^\mu)^{\dot\alpha\alpha} & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} \delta^\alpha_\beta & 0 \\ 0 & \delta^{\dot\alpha}_{\dot\beta} \end{pmatrix}, \frac{1}{2}\Sigma^{\mu\nu} = \begin{pmatrix} i(\sigma^{\mu\nu})_\alpha^{\ \beta} & 0 \\ 0 & i(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \end{pmatrix}$$

with

$$(\sigma^\mu)_{\alpha\dot\alpha} = (I_2, \vec\sigma), (\bar\sigma^\mu)^{\dot\alpha\alpha} = (I_2, -\vec\sigma)$$

where $\vec\sigma = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauil matrices. Furthermore,

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu), (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu).$$

Some spinor relations:

- $\chi\psi := \chi^\alpha\psi_\alpha = \psi\chi$, $\bar\chi\bar\psi := \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\psi\bar\chi$, $(\chi\psi)^\dagger = \psi^\dagger\chi^\dagger = \bar\psi\bar\chi$
- $\chi\sigma^\mu\bar\psi := \chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\psi^{\dot\alpha}$, $\bar\chi\bar\sigma^\mu\psi := \bar\chi_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\alpha}\psi_\alpha$
- $(\chi\sigma^\mu\bar\psi)^\dagger = \psi\sigma^\mu\bar\chi$
- $\chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi$
- $\chi\sigma^{\mu\nu}\psi = -\psi\sigma^{\mu\nu}\chi$, $\bar\chi\bar\sigma^{\mu\nu}\bar\psi = -\bar\psi\bar\sigma^{\mu\nu}\bar\chi$
- $(\chi\sigma^{\mu\nu}\psi)^\dagger = \bar\chi\bar\sigma^{\mu\nu}\bar\psi$

Problem 1: Variations on the SUSY algebra

a) Write down the ($N = 1$) SUSY algebra in terms of the 10 bosonic generators of the Poincaré group (P^μ, $J^{\mu\nu}$) and the 4 fermionic generators (Q_α, and $\bar{Q}_{\dot\alpha}$). Here Q_α ($\alpha = 1, 2$), and $\bar{Q}_{\dot\alpha}$ ($\dot\alpha = \dot 1, \dot 2$) are 2-component Weyl spinors transforming according to the $(1/2, 0)$ and $(0, 1/2)$-representations of the Lorentz group, respectively.

b) In 4-component notation we define the Majorana spinor $Q_{M,a}$ ($a = 1, 2, \dot 1, \dot 2$):

$$Q_M = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix} .$$

Show that the Dirac-adjoint spinor is given by $\bar{Q}_M = (Q^\beta, \bar{Q}_{\dot\beta})$.

c) Show that

$$\begin{aligned} [P^\mu, Q_M] &= 0\,, \\ [J^{\mu\nu}, Q_M] &= -\frac{1}{2}\,\Sigma^{\mu\nu}\, Q_M\,, \\ \{Q_M, \bar{Q}_M\} &= 2\gamma^\mu P_\mu \end{aligned}$$

with γ^μ and $\Sigma^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ in the Weyl representation.
Hint: Consider $\{Q_{M,a}, \bar{Q}_{M,b}\}$ for fixed $a, b = 1, 2, \dot 1, \dot 2$.

d) The SUSY algebra can be expressed entirely in terms of commutators with the help of anti-commuting parameters ξ^α, $\bar\xi_{\dot\alpha}$ and η^α, $\bar\eta_{\dot\alpha}$:

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = \ldots = [P_\mu, \xi^\alpha] = 0\,.$$

Evaluate the following commutators using the algebra in a):

(i) $[P_\mu, \xi Q] = [P_\mu, \bar\xi\bar Q]$

(ii) $[J^{\mu\nu}, \xi Q]$, $[J^{\mu\nu}, \bar\xi\bar Q]$

(iii) $[\xi Q, \eta Q] = [\bar\xi\bar Q, \bar\eta\bar Q]$

(iv) $[\xi Q, \bar\eta\bar Q]$

where we use the summation convention $\xi Q := \xi^\alpha Q_\alpha$, $\bar\xi\bar Q := \bar\xi_{\dot\alpha}\bar Q^{\dot\alpha}$.

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Problem 2: A supersymmetric Lagrangian

a) Write down the Wess-Zumino Lagrangian for massless ($m = 0$) and non-interacting ($g = 0$) fields A and ψ

b) In this case ($m = 0, g = 0$) the SUSY transformations are given by

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\,\xi\psi\,, \\ \delta_\xi \psi &= -i\sqrt{2}\,\sigma^\mu \bar{\xi}\,\partial_\mu A \end{aligned}$$

Here, we use again the summation convention and free indices α have been suppressed. Write down the transformation rules for the conjugate fields which can be obtained by hermitian conjugation:

(i) $\delta_\xi A^* = (\delta_\xi A)^\dagger = \ldots$

(ii) $\delta_\xi \bar{\psi} = (\delta_\xi \psi)^\dagger = \ldots$

c) Show that

$$\frac{1}{\sqrt{2}}\delta_\xi\left[(\partial_\mu A^*)(\partial^\mu A)\right] = -\bar{\xi}\bar{\psi}\,\Box A - \xi\psi\,\Box A^* + \partial_\mu K_1^\mu$$

with $\Box := \partial_\mu \partial^\mu$. Specify K_1^μ.

d) Show that

$$\frac{1}{\sqrt{2}}\delta_\xi\left[i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi\right] = \bar{\xi}\bar{\psi}\,\Box A + \xi\psi\,\Box A^* + \partial_\mu K_2^\mu$$

Specify K_2^μ.
Hint: $\sigma^\mu\bar{\sigma}^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu}$, $\bar{\sigma}^\mu\sigma^\nu = \eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu}$ and $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$, $\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\nu\mu}$.

e) Conclude that the action is invariant under this symmetry: $\delta_\xi S = 0$.

1) a)

$$[P^\mu, P^\nu] = 0$$

$$[J^{\mu\nu}, P^\rho] = i(\eta^{\mu\rho} P^\nu - \eta^{\nu\rho} P^\mu)$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = -i(\eta^{\mu\rho} J^{\nu\sigma} - \eta^{\mu\sigma} J^{\nu\rho} - (\mu \leftrightarrow \nu))$$

$$[P^\mu, Q_\alpha] = 0 = [P^\mu, \bar{Q}^{\dot\beta}]$$

$$[J^{\mu\nu}, Q_\alpha] = -i\,(\sigma^{\mu\nu})_\alpha{}^\beta\, Q_\beta$$

$$[J^{\mu\nu}, \bar{Q}^{\dot\alpha}] = -i\,(\bar\sigma^{\mu\nu})^{\dot\alpha}{}_{\dot\beta}\, \bar{Q}^{\dot\beta}$$

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2\,\sigma^\mu{}_{\alpha\dot\beta} P_\mu$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$$

b)

$$Q_M = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\alpha} \end{pmatrix}, \quad \bar{Q}_M = Q_M^\dagger \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = (\bar{Q}_{\dot\alpha} \;\; Q^\alpha) \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} = (Q^\alpha \;\; \bar{Q}_{\dot\alpha})$$

c)

$$[P_\mu, (Q_M)_a] = 0$$

$$[J^{\mu\nu}, Q_M] = \begin{pmatrix} [J^{\mu\nu}, Q_\alpha] \\ [J^{\mu\nu}, \bar{Q}^{\dot\alpha}] \end{pmatrix} = -\begin{pmatrix} i\sigma^{\mu\nu} & 0 \\ 0 & i\bar\sigma^{\mu\nu} \end{pmatrix} \begin{pmatrix} Q_\beta \\ \bar{Q}^{\dot\beta} \end{pmatrix} = -\frac{1}{2} \Sigma^{\mu\nu} Q_M$$

$$\{Q_{M_a}, \bar{Q}_{M_b}\} = \begin{pmatrix} \{Q_\alpha, Q^\beta\} & \{Q_\alpha, \bar{Q}_{\dot\beta}\} \\ \{\bar{Q}^{\dot\alpha}, Q^\beta\} & \{\bar{Q}^{\dot\alpha}, \bar{Q}_{\dot\beta}\} \end{pmatrix} = \begin{pmatrix} 0 & 2\sigma^\mu{}_{\alpha\dot\beta} P_\mu \\ 2\bar\sigma^{\mu\dot\alpha\beta} P_\mu & 0 \end{pmatrix}$$

$$= 2\,\gamma^\mu P_\mu$$

d)

(i) $[P_\mu, \xi Q] = \xi^\alpha [P_\mu, Q_\alpha] = 0 = [P_\mu, \bar\xi \bar{Q}]$ (or use: $[A, BC] = B[A, C] + [A, B]C$ mais attention avec anti-commutateurs !!)

(ii) $[J^{\mu\nu}, \xi Q] = \xi^\alpha [J^{\mu\nu}, Q_\alpha] = -i\, \xi \sigma^{\mu\nu} Q$

$$[J^{\mu\nu}, \bar\xi \bar{Q}] = \bar\xi_{\dot\alpha} [J^{\mu\nu}, \bar{Q}^{\dot\alpha}] = -i\, \bar\xi \bar\sigma^{\mu\nu} \bar{Q}$$

(iii) $[\xi Q, \eta Q] = \xi^\alpha Q_\alpha \eta^\beta Q_\beta - \eta^\beta Q_\beta \xi^\alpha Q_\alpha = -\xi^\alpha Q_\alpha Q_\beta \eta^\beta - \xi^\alpha Q_\beta Q_\alpha \eta^\beta$

$$= -\xi^\alpha \underbrace{\{Q_\alpha, Q_\beta\}}_{=0} \eta^\beta = 0$$

Similarly $[\bar\xi \bar{Q}, \bar\eta \bar{Q}] = 0$

(iv) $[\xi Q, \bar\eta \bar{Q}] = [\xi Q, \bar{Q} \bar\eta] = \xi^\alpha \{Q_\alpha, \bar{Q}_{\dot\beta}\} \bar\eta^{\dot\beta} = 2\, \xi \sigma^\mu \bar\eta\, P_\mu$

$$= \xi^\alpha Q_\alpha \bar{Q}_{\dot\beta} \bar\eta^{\dot\beta} - \bar{Q}_{\dot\beta} \bar\eta^{\dot\beta} \xi^\alpha Q_\alpha = \xi^\alpha (Q_\alpha \bar{Q}_{\dot\beta} + \bar{Q}_{\dot\beta} Q_\alpha) \bar\eta^{\dot\beta}$$

NR:

$$\{\bar{Q}^{\dot\alpha}, Q^\beta\} = \{\epsilon^{\dot\alpha\dot\beta}\bar{Q}_{\dot\beta}, \epsilon^{\beta\alpha} Q_\alpha\}$$

$$= \epsilon^{\dot\alpha\dot\beta}\epsilon^{\beta\alpha}\{\bar{Q}_{\dot\beta}, Q_\alpha\} = \epsilon^{\dot\alpha\dot\beta}\epsilon^{\beta\alpha}\{Q_\alpha, \bar{Q}_{\dot\beta}\}$$

$$= \epsilon^{\dot\alpha\dot\beta}\epsilon^{\beta\alpha}\, 2\sigma^\mu_{\alpha\dot\beta} P_\mu \qquad , \epsilon^{\dot\alpha\dot\beta} = i\sigma^2 = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \epsilon^{\alpha\beta}$$

$$\cancel{= 2\,\epsilon^{\dot\alpha\dot\beta}\epsilon^{\beta\alpha}\sigma^{\mu T}_{\dot\beta\alpha} P_\mu}$$

$$\cancel{= 2\,\epsilon^{\dot\alpha\dot\beta}\sigma^{\mu T}_{\dot\beta\alpha}\epsilon^{\alpha\beta} P_\mu}$$

$$\cancel{= 2\,\underbrace{(\epsilon\,\sigma^{\mu T}\epsilon)}^{\dot\alpha\beta} P_\mu}$$

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \sigma^1$$

$$\cancel{= (\mathbb{1}, i^2\,\underbrace{\sigma^2\sigma^{\mu T}\sigma^2}_{= -\sigma^2\vec\sigma^T\sigma^2 = \vec\sigma})}$$

$$= -2\,\epsilon^{\dot\alpha\dot\beta}\sigma^{\mu T}_{\dot\beta\alpha}\epsilon^{\alpha\beta} P_\mu$$

$$= -2\,\underbrace{(\epsilon\,\sigma^{\mu T}\epsilon)}_{i\sigma^2\sigma^{\mu T}i\sigma^2}{}^{\dot\alpha\beta} P_\mu = -2\,(-\bar\sigma^\mu)^{\dot\alpha\beta} P_\mu = 2\,\bar\sigma^{\mu\,\dot\alpha\beta} P_\mu$$

$$= -\begin{cases}\mathbb{1}_2\\ \sigma^2\vec\sigma^T\sigma^2 = -\vec\sigma\end{cases}$$

2a) $m=0, g=0$:

$$\mathcal{L} = (\partial_\mu A^*)(\partial^\mu A) + i\bar\psi\bar\sigma^\mu\partial_\mu\psi$$

b)

$$\delta_\xi A = \sqrt{2}\,\xi\psi$$

$$\delta_\xi \psi = -i\sqrt{2}\,\sigma^\mu\bar\xi\,\partial_\mu A$$

$$\Rightarrow \delta_\xi A^* = \sqrt{2}\,\bar\xi\bar\psi$$

$$\delta_\xi\bar\psi = i\sqrt{2}\,\xi\sigma^\mu\,\partial_\mu A^*$$

Hermitian! $(\sigma^{\mu\dagger} = \sigma^\mu \ \forall\mu)$

$$\sigma^\mu\bar\sigma^\nu = g^{\mu\nu} + 2\sigma^{\mu\nu}$$

$$\sigma^{\mu\nu} = -\sigma^{\nu\mu}$$

$$\bar\sigma^\mu\sigma^\nu = g^{\mu\nu} + 2\bar\sigma^{\mu\nu}$$

Show that $\delta_\xi\mathcal{L} = \partial_\mu K^\mu$

c)

* $$\delta_\xi\big((\partial_\mu A^*)(\partial^\mu A)\big) = (\partial_\mu\delta_\xi A^*)(\partial^\mu A) + (\partial_\mu A^*)(\partial^\mu\delta_\xi A)$$

$$= \sqrt{2}\,\{(\partial_\mu\bar\xi\bar\psi)(\partial^\mu A) + (\partial_\mu A^*)(\partial^\mu\xi\psi)\}$$

$$= \sqrt{2}\,\{\partial_\mu[\bar\xi\bar\psi\,\partial^\mu A] - g^{\mu\lambda}\bar\xi\bar\psi\,\partial_\mu\partial_\lambda A + \partial_\mu[\xi\psi\,\partial^\mu A^*] - g^{\mu\lambda}\xi\psi\,\partial_\mu\partial_\lambda A^*\} = \sqrt{2}\,\{\partial_\mu\overbrace{[\bar\xi\bar\psi\,\partial^\mu A + \xi\psi\,\partial^\mu A^*]}^{K_1^\mu} - \bar\xi\bar\psi\,\Box A - \xi\psi\,\Box A^*\}$$

d)

* $$\delta_\xi(i\bar\psi\bar\sigma^\mu\partial_\mu\psi) = i\big((\delta_\xi\bar\psi)\bar\sigma^\mu\partial_\mu\psi + \bar\psi\bar\sigma^\mu\partial_\mu\delta_\xi\psi\big)$$

$$= i\sqrt{2}\,\big([i\xi\sigma^\lambda\partial_\lambda A^*]\bar\sigma^\mu\partial_\mu\psi + \bar\psi\bar\sigma^\mu\partial_\mu[-i\sigma^\lambda\bar\xi\,\partial_\lambda A]\big)$$

$$= -\sqrt{2}\,\{\xi\sigma^\lambda\bar\sigma^\mu(\partial_\mu\psi)(\partial_\lambda A^*) - \bar\psi\underbrace{\bar\sigma^\mu\sigma^\lambda}_{g^{\lambda\mu}+2\sigma^{\lambda\mu}}\bar\xi\,\partial_\mu\partial_\lambda A\}$$

$$\overbrace{\sigma^{\lambda\mu}}^{A}\,\overbrace{\partial_\mu\partial_\lambda}^{S} = 0$$

$$= -\sqrt{2}\,\{\partial_\mu(\xi\sigma^\lambda\bar\sigma^\mu\psi\,\partial_\lambda A^*) - \xi\sigma^\lambda\bar\sigma^\mu\psi\,\partial_\mu\partial_\lambda A^* - \bar\psi\bar\sigma^\mu\sigma^\lambda\bar\xi\,\partial_\mu\partial_\lambda A\}$$

$$\tfrac{1}{2}\bar\psi\,(\underbrace{\bar\sigma^\mu\sigma^\lambda + \sigma^\lambda\bar\sigma^\mu}_{2g^{\mu\lambda}})\bar\xi\,\partial_\mu\partial_\lambda A$$

$$= -\sqrt{2}\,\{\partial_\mu(\overbrace{\xi\sigma^\lambda\bar\sigma^\mu\psi\,\partial_\lambda A^*}^{-K_2^\mu}) - \xi\psi\,\Box A^* - \bar\xi\bar\psi\,\Box A\}$$

e)

$$\Rightarrow \delta_\xi\mathcal{L} = \partial_\mu K^\mu, \qquad K^\mu = \sqrt{2}\,\{\bar\xi\bar\psi\,\partial^\mu A + \xi\psi\,\partial^\mu A^* - \xi\sigma^\lambda\bar\sigma^\mu\psi\,\partial_\lambda A^*\}$$

total divergence

$$\Rightarrow \delta_\xi S = 0$$