M2R PSA
Physics beyond the standard model
All documents allowed

03 février 2010, durée 3h00

PART I: Supersymmetry

Hints:

- $\eta^{\mu\nu} = \mathrm{diag}[1, -1, -1, -1]$ is the (flat) space-time metric.
- $\sigma^{\mu} = (I_2, \vec{\sigma})$, $\bar{\sigma}^{\nu} = (I_2, -\vec{\sigma})$ with $I_2 = \mathrm{diag}[1, 1]$ and the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \sigma^2 = \begin{pmatrix} 0 & \text{-i} \\ \text{i} & 0 \end{pmatrix} , \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix}$$

- $\sigma^i = \sigma^{i\dagger}$ (Hermitian)
- $\mathrm{Tr}\, \sigma^i = 0$ (Traceless)
- $\{\sigma^i, \sigma^j\} = 2\ \delta^{ij}\ I_2$ (Clifford algebra)
- $[\sigma^i, \sigma^j] = 2\ i\epsilon^{ijk}\sigma^k$ (Lie algebra)
- $\sigma^i\sigma^j = \frac{1}{2}\ \{\sigma^i, \sigma^j\} + \frac{1}{2}\ [\sigma^i, \sigma^j] = \delta^{ij} I_2 + i\epsilon^{ijk}\sigma^k$

Problem 1: The supersymmetric ground state

a) Show that $\mathrm{Tr}(\sigma^\mu \bar{\sigma}^\nu) \equiv \sigma^\mu_{\alpha\dot{\beta}}\, \bar{\sigma}^{\nu\dot{\beta}\alpha} = 2\,\eta^{\mu\nu}$.

b) Show that $\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}\, \bar{\sigma}^{\nu\dot{\beta}\alpha} = 4P^\nu$.

c) Show that the operator $H := P^0$ has real and non-negative eigenvalues $E \geq 0$.

d) If $|0\rangle$ is the ground state (vacuum state) show that

$$\langle 0|H|0\rangle = 0 \quad \Leftrightarrow \quad Q_\alpha\,|0\rangle = \bar{Q}_{\dot{\alpha}}\,|0\rangle = 0 \quad (\alpha, \dot{\alpha} = 1, 2)\,.$$

Conclusion: A ground state with positive energy breaks supersymmetry spontaneously: $[H, Q_\alpha] = 0$ (SUSY algebra) but $Q_\alpha\,|0\rangle \neq 0$.

Problem 2: Number of bosonic and fermionic degrees of freedom in SUSY multiplets

Recall the Casimir operators of the Poincaré algebra, P^2 and W^2 where W_μ is the Pauli-Lubanski vector. Note that: $[P^2, Q_\alpha] = [P^2, \bar{Q}_{\dot{\alpha}}] = 0$ but $[W^2, Q_\alpha] \neq 0$, $[W^2, \bar{Q}_{\dot{\alpha}}] \neq 0$. Thus, irreducible (and therefore also reducible) representations of the supersymmetry algebra will contain states with different spins. Schematically we can write

$$Q_\alpha\,|B\rangle = |F\rangle\ ,\ Q_\alpha\,|F\rangle = |B\rangle\ ,$$

where $|B\rangle$ is a bosonic and $|F\rangle$ a fermionic state.

Definition: $(-1)^{N_F}$ is an operator defined such that

$$(-1)^{N_F}\,|B\rangle = +\,|B\rangle\ ,\ (-1)^{N_F}\,|F\rangle = -\,|F\rangle\ .$$

a) Show that $Q_\alpha(-1)^{N_F} = -(-1)^{N_F} Q_\alpha$.

b) Show that $\mathrm{Tr}[(-1)^{N_F}] = 0$ (for fixed non-zero P_μ) where the trace takes all states of the representation/multiplet into account. (Hint: Evaluate $\mathrm{Tr}[(-1)^{N_F}\{Q_\alpha, \bar{Q}_{\dot{\beta}}\}]$ directly and by using the right side of the corresponding supersymmetry algebra relation.)

c) Conclude that every representation of the supersymmetry algebra contains an equal number of bosonic and fermionic states.

1)

a)

$$\mathrm{Tr}(\sigma^\mu \bar\sigma^\nu) \equiv \sigma^\mu{}_{\alpha\dot\beta}\, \bar\sigma^{\nu\dot\beta\alpha} \stackrel{!}{=} 2\eta^{\mu\nu}$$

- $\mu=\nu=0$: $\mathrm{Tr}(I_2) = 2 = 2\eta^{00}$ ✓
- $\mu=\nu=i$: $-\mathrm{Tr}(\sigma^{i2}) = -\mathrm{Tr}(I_2) = -2 = 2\eta^{ii}$ ✓
- $\mu=0, \nu=i$: $-\mathrm{Tr}(\sigma^i) = 0 = 2\eta^{0i}$ ✓
- $\mu=i, \nu=0$: $+\mathrm{Tr}(\sigma^i) = 0 = 2\eta^{i0}$ ✓
- $\mu=i, \nu=j$: $-\mathrm{Tr}(\sigma^i\sigma^j) \underset{\text{Hint}}{=} -\mathrm{Tr}(\delta^{ij} I_2) + i\epsilon^{ij}{}_k \cancelto{0}{\mathrm{Tr}(\sigma^k)}$

$$= -2\delta^{ij} = 2\eta^{ij}$$

Together: $\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = 2\eta^{\mu\nu}$ ☒

b)

SUSY algebra (N=1): $\{Q_\alpha, \bar Q_{\dot\beta}\} = 2\sigma^\mu{}_{\alpha\dot\beta} P_\mu$

$$\Rightarrow \{Q_\alpha, \bar Q_{\dot\beta}\}\bar\sigma^{\nu\dot\beta\alpha} = 2\underbrace{\sigma^\mu{}_{\alpha\dot\beta}\bar\sigma^{\nu\dot\beta\alpha}}_{2\eta^{\mu\nu}} P_\mu = 4P^\nu$$

☒

c)

$H := P^0$

$$\Rightarrow H = \frac{1}{4}\{Q_\alpha, \bar Q_{\dot\beta}\}\bar\sigma^{0\dot\beta\alpha} = \frac{1}{4}\left(Q_1\bar Q_{\dot 1} + Q_2\bar Q_{\dot 2} + \bar Q_{\dot 1}Q_1 + \bar Q_{\dot 2}Q_2\right)$$

$$\bar Q_{\dot\alpha} = (Q_\alpha)^\dagger$$

$$\Rightarrow H = \frac{1}{4}\left(Q_1Q_1^\dagger + Q_1^\dagger Q_1 + Q_2Q_2^\dagger + Q_2^\dagger Q_2\right)$$

$$= \frac{1}{4}\Big(\underbrace{(Q_1+Q_1^\dagger)^2}_{\geq 0} + \underbrace{(Q_2+Q_2^\dagger)^2}_{\geq 0}\Big) \qquad \left[Q_1^2 = 0,\ Q_2^2 = 0\right]$$

☒

- $H = H^\dagger \Rightarrow H$ has real eigenvalues
- Eigenvalues of $(Q_i+Q_i^\dagger)^2$ are the square of the eigenvalues of $Q_i+Q_i^\dagger$ and therefore non-negative

d)

$$\langle 0 | H | 0 \rangle = \frac{1}{4} \langle 0 | (\underbrace{Q_1 Q_1^\dagger + Q_1^\dagger Q_1}_{\geq 0} + \underbrace{Q_2 Q_2^\dagger + Q_2^\dagger Q_2}_{\geq 0}) | 0 \rangle = 0$$

$\Leftrightarrow$ $\langle 0 | (Q_1 Q_1^\dagger + Q_1^\dagger Q_1) | 0 \rangle = 0$ and $\langle 0 | Q_2 Q_2^\dagger + Q_2^\dagger Q_2 | 0 \rangle = 0$

$\Leftrightarrow$ $\| Q_1^\dagger |0\rangle \|^2 = 0$ $\| Q_1 |0\rangle \|^2 = 0$ $\| Q_2^\dagger |0\rangle \|^2 = 0$ $\| Q_2 |0\rangle \|^2 = 0$

$\Leftrightarrow$ $Q_\alpha |0\rangle = \bar{Q}_{\dot{\alpha}} |0\rangle = 0$

2)

a)

$$Q_\alpha (-1)^{N_F} |B\rangle = Q_\alpha |B\rangle = |F\rangle$$

$$-(-1)^{N_F} Q_\alpha |B\rangle = -(-1)^{N_F} |F\rangle = |F\rangle$$

$$Q_\alpha (-1)^{N_F} |F\rangle = -Q_\alpha |F\rangle = -|B\rangle$$

$$-(-1)^{N_F} Q_\alpha |F\rangle = -(-1)^{N_F} |B\rangle = -|B\rangle$$

$$\Rightarrow \quad Q_\alpha (-1)^{N_F} = -(-1)^{N_F} Q_\alpha$$

$\boxtimes$

b)

$$\mathrm{Tr}\left[(-1)^{N_F}\{Q_\alpha, \bar{Q}_{\dot\beta}\}\right] = \mathrm{Tr}\left[(-1)^{N_F}\left(Q_\alpha \bar{Q}_{\dot\beta} + \bar{Q}_{\dot\beta} Q_\alpha\right)\right]$$

$$= \mathrm{Tr}\left[\underbrace{(-1)^{N_F} Q_\alpha}_{2\,a)} \bar{Q}_{\dot\beta}\right] + \mathrm{Tr}\left[\underbrace{(-1)^{N_F} \bar{Q}_{\dot\beta} Q_\alpha}_{\text{cyclicity of trace}}\right]$$

$$= -\mathrm{Tr}\left[Q_\alpha (-1)^{N_F} \bar{Q}_{\dot\beta}\right] + \mathrm{Tr}\left[Q_\alpha (-1)^{N_F} \bar{Q}_{\dot\beta}\right] = 0$$

$$= \mathrm{Tr}\left[(-1)^{N_F}\, 2\sigma^\mu{}_{\alpha\dot\beta} P_\mu\right] = 2\sigma^\mu{}_{\alpha\dot\beta}\, \mathrm{Tr}\left[(-1)^{N_F} P_\mu\right]$$

For $P_\mu \neq 0 \Rightarrow \mathrm{Tr}\left[(-1)^{N_F}\right] = 0$

$\boxtimes$

c)

$$\langle B| (-1)^{N_F} |B\rangle = +1, \quad \langle F| (-1)^{N_F} |F\rangle = -1$$

$$\Rightarrow \quad \mathrm{Tr}\left[(-1)^{N_F}\right] = \sum_{\substack{\text{ONB}\\ \text{of states}}} \langle\psi| (-1)^{N_F} |\psi\rangle = n_B \cdot 1 + n_F \cdot (-1)$$

$$= n_B - n_F = 0$$

$\Rightarrow \quad n_B = n_F$

$\boxtimes$