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Lecture Notes

Supersymmetry: Theory & Application

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Contents

1 Preliminaries

- Akın Wingerter (akin@lpsc.in2p3.fr), substituting Ingo Schienbein
- Postdoc here at LPSC; working on particle physics (Higgs, SUSY, GUTs, neutrinos, in distant past also extra dimensions and string theory)
- Date/time/place for lecture:

1st lecture:	2 hours	Thursday, 24.01.2013, 13:30h
2nd lecture:	3 hours	Thursday, 31.01.2013, 13:30h
3rd lecture:	3 hours	Thursday, 07.02.2013, 13:30h

  Here at the LPSC!
- Language of instruction: English

  If you have problems understanding me (because I might be talking too fast or you may not catch my accent), please tell me!!!

  You can ask questions in English (preferred) or French. If I do not understand your question, Ingo will translate.

  Do not hesitate to ask any question! There are no stupid questions, only stupid answers, and answering is my job!
- Final exam questions will be based on (optional) homework problems that are interspersed in the lecture
- There is a web page for this course (not up yet), please check it out:

  http://lpsc.in2p3.fr/wingerter/
- After each lecture, I will ask you to fill out an (anonymous) online survey so that I get some feedback. Please participate!
- Please give me your email addresses.
- Literature:
  - Maggiore [2]: Excellent book on QFT
  - Martin [3]: SUSY; covers everything from motivation to calculation to phenomenology; maybe a bit too difficult for master-II
  - Kalka and Soff [1]: SUSY; very explicit calculations; unfortunately in German
  - Muller-Kirsten and Wiedemann [4]: SUSY; even more explicit, but horrible type-setting (handwritten formulae); in English
- Goal for today: Reach the Standard Model! Will not rush, profound understanding more important than meeting goals.

2 The Standard Model

2.1 One-page Summary of the World

Gauge group

$$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$$

Particle content

MATTER				HIGGS		GAUGE	
$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3},\mathbf{2})_{1/3}$	$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_{-1}$	$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_1$	A	$(\mathbf{1},\mathbf{1})_0$
u_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{-4/3}$	e_R^c	$(\mathbf{1},\mathbf{1})_2$			W	$(\mathbf{1},\mathbf{3})_0$
d_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{2/3}$	ν_R^c	$(\mathbf{1},\mathbf{1})_0$			G	$(\mathbf{8},\mathbf{1})_0$

Lagrangian (Lorentz + gauge + renormalizable)

$$\mathcal{L} = -\frac{1}{4} G^{\alpha}_{\mu\nu} G^{\alpha\mu\nu} + \ldots \overline{Q}_k \not{D} Q_k + \ldots (D_\mu H)^\dagger (D^\mu H) - \mu^2 H^\dagger H - \frac{\lambda}{4!}(H^\dagger H)^2 + \ldots Y_{k\ell} \overline{Q}_k H (u_R)_\ell$$

Spontaneous symmetry breaking

- $H \to H' + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$
- $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_Q$
- $A, W^3 \to \gamma, Z^0$ and $W^1_\mu, W^2_\mu \to W^+, W^-$
- Fermions acquire mass through Yukawa couplings to Higgs

2.2 Filling in the Details

2.2.1 The Particle Content

Matter				Higgs		Gauge	
$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3},\mathbf{2})_{1/3}$	$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_{-1}$	$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_1$	A	$(\mathbf{1},\mathbf{1})_0$
u_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{-4/3}$	e_R^c	$(\mathbf{1},\mathbf{1})_2$			W	$(\mathbf{1},\mathbf{3})_0$
d_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{2/3}$	ν_R^c	$(\mathbf{1},\mathbf{1})_0$			G	$(\mathbf{8},\mathbf{1})_0$

$Q^c = \begin{pmatrix} u_L^c \\ d_L^c \end{pmatrix}$	$(\overline{\mathbf{3}},\overline{\mathbf{2}})_{-1/3}$	$L^c = \begin{pmatrix} \nu_L^c \\ e_L^c \end{pmatrix}$	$(\mathbf{1},\overline{\mathbf{2}})_1$	$H = \begin{pmatrix} h^- \\ h^0 \end{pmatrix}$	$(\mathbf{1},\overline{\mathbf{2}})_{-1}$	A	$(\mathbf{1},\mathbf{1})_0$
u_R	$(\mathbf{3},\mathbf{1})_{4/3}$	e_R	$(\mathbf{1},\mathbf{1})_{-2}$			W	$(\mathbf{1},\mathbf{3})_0$
d_R	$(\mathbf{3},\mathbf{1})_{-2/3}$	ν_R	$(\mathbf{1},\mathbf{1})_0$			G	$(\mathbf{8},\mathbf{1})_0$

Nota bene:

- Since the SM is *chiral*, we work with 2-component *Weyl spinors*.
- *Chiral* means that the left-handed and the right-handed particles do not transform differently under the gauge group: E.g. $u_L \sim (\mathbf{3},\mathbf{2})_{1/3}$ and $u_R \sim (\mathbf{3},\mathbf{1})_{4/3}$
- For every particle, there is an anti-particle which is usually not explicitly listed.
- Note that u_R^c is the charge conjugate of a right-handed particle and as such transforms as a left-handed particle. More precisely, one should write $(u_R)^c$. Some other common notation: u_L^c (for $(u^c)_L$), $\overline{u}$ or simply u or U.
- The reason why we list e.g. u_R^c instead of u_R is that we want to use only *left-handed* particles (important later for SUSY).
- The doublet structure of e.g. $Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$ indicates how it transforms under $\mathrm{SU}(2)_L$. It has absolutely nothing to do with Dirac spinors.
- The right-handed neutrino ν_R is a hypotetical particle whose existence has not been established. I am including it only for later reference when I will talk about GUTs and the seesaw mechanism.

- In SU(2), the representations $\mathbf{2}$ and $\overline{\mathbf{2}}$ are *equivalent*, but not the identical/equal/same! If one wants to replace $\overline{\mathbf{2}}$ by $\mathbf{2}$, one needs some extra work.

Homework 2.1 *Let χ be a left-handed Weyl spinor. Show that $\eta := i\sigma_2\chi^*$ transforms as a right-handed Weyl-spinor. Here, σ_2 is the second Pauli matrix.*

Hint: Since χ is left-handed, it will transform under the Lorentz group as $\chi \to \Lambda_L \chi$. You need to show that η transforms under the Lorentz group as a right-handed Weyl spinor, i.e. $\eta \to \Lambda_R \eta$. You can find the explicit form of Λ_L and Λ_R in Maggiore [2], but for this homework just use the identity $\sigma_2 \Lambda_L^* \sigma_2 = \Lambda_R$.

2.2.2 How to build a Lorentz scalar

Scalars: Spin 0

Real field ϕ

$$\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2 \tag{2.1}$$

Complex field $\phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$

$$\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{2.2}$$

Note that Eq. (2.2) has a U(1) symmetry. If $\phi \to e^{i\alpha}\phi$, we have:

$$\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \;\to\; \partial_\mu (e^{i\alpha}\phi)^* \partial^\mu (e^{i\alpha}\phi) - m^2 (e^{i\alpha}\phi)^* (e^{i\alpha}\phi) \;=\; \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

Complex (Higgs!) doublet $\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \varphi_1 + i\varphi_2 \\ \varphi_3 + i\varphi_4 \end{pmatrix}$

$$\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \tag{2.3}$$

Note that Eq. (2.3) is invariant under SU(2). If $\phi \to e^{i(\alpha_1\sigma_1 + \alpha_2\sigma_2 + \alpha_3\sigma_3)}\phi$, where $\alpha_i \in \mathbb{R}$ are arbitrary real numbers and σ_1, σ_2, σ_3 are the Pauli matrices, then:

$$\begin{aligned}
\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi \;&\to\; \partial_\mu (e^{i\alpha_i\sigma_i}\phi)^\dagger \partial^\mu (e^{i\alpha_i\sigma_i}\phi) - m^2 (e^{i\alpha_i\sigma_i}\phi)^\dagger (e^{i\alpha_i\sigma_i}\phi) \\
&= \partial_\mu \left[\phi^\dagger (e^{i\alpha_i\sigma_i})^\dagger\right] \partial^\mu (e^{i\alpha_i\sigma_i}\phi) - m^2 (\phi^\dagger (e^{i\alpha_i\sigma_i})^\dagger)(e^{i\alpha_i\sigma_i}\phi) \quad \Big| (AB)^\dagger = B^\dagger A^\dagger \\
&= \partial_\mu \phi^\dagger \left[(e^{i\alpha_i\sigma_i})^\dagger e^{i\alpha_i\sigma_i}\right] \partial^\mu \phi - m^2 \left[\phi^\dagger (e^{i\alpha_i\sigma_i})^\dagger e^{i\alpha_i\sigma_i}\right] \phi \\
&= \partial_\mu \phi^\dagger \left[(e^{-i\alpha_i\sigma_i^\dagger}) e^{i\alpha_i\sigma_i}\right] \partial^\mu \phi - m^2 \left[\phi^\dagger (e^{-i\alpha_i\sigma_i^\dagger} e^{i\alpha_i\sigma_i}\right] \phi \\
&= \partial_\mu \phi^\dagger \left[(e^{-i\alpha_i\sigma_i}) e^{i\alpha_i\sigma_i}\right] \partial^\mu \phi - m^2 \left[\phi^\dagger (e^{-i\alpha_i\sigma_i} e^{i\alpha_i\sigma_i}\right] \phi \quad \Big| \sigma_i^\dagger = \sigma_i \\
&= \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi \quad \Big| e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]} \rightsquigarrow e^A e^{-A} = \mathbb{1}
\end{aligned}$$

Fermions: Spin 1/2

Left-handed Weyl spinor

$$i\psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L \tag{2.4}$$

Right-handed Weyl spinor

$$i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R \tag{2.5}$$

Mass term mixes left and right

$$i\psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \tag{2.6}$$

This will be of paramount importance later in the SM, so do not forget this point!

Dirac spinor in chiral basis

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{2.7}$$

We can now rewrite Eq. (2.8) (into the familiar form) as

$$i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\overline{\Psi}\Psi \quad \text{with} \quad \overline{\Psi} = \Psi^\dagger \gamma^0 \quad \text{and} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \tag{2.8}$$

Note that it is more "natural" to write down the SM with Weyl spinors, because

- weak interactions distinguish between left- and right-handed particles,
- (the need for) the Higgs mechanism is easier to understand,
- Weyl spinors are the basic "building blocks" (smallest irreps of Lorentz group).

Vector Bosons: Spin 1

U(1) gauge boson ("Photon")

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.9}$$

Mass term in SM forbidden by gauge symmetry, but in principle allowed (e.g. by Lorentz invariant)

In principle, there is a second invariant

$$-\frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} \quad \text{with} \quad \widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} \tag{2.10}$$

Relevant for SU(3) $\rightsquigarrow$ strong CP problem (not clear why it is not present in SM)

Kinetic mixing, if there are two Abelian gauge groups, $\mathrm{U}(1)_A$ and $\mathrm{U}(1)_B$

$$-\frac{1}{4}F_{A\mu\nu}F_A^{\mu\nu} - \frac{1}{4}F_{B\mu\nu}F_B^{\mu\nu} - \frac{1}{4}F_{A\mu\nu}F_B^{\mu\nu} \tag{2.11}$$

SU(2) gauge bosons will be discussed after the concept of covariant derivative has been introduced.

2.2.3 Gauge Symmetries

Idea: Generate dynamics (i.e. interactions) from free Lagrangian by imposing local (i.e. now $\alpha = \alpha(x)$) symmetries.

Does not fall from heavens; generalization of "minimal coupling" in electrodynamics/quantum mechanics.

Final judge is experiment: It works!

Local Gauge Invariance for Complex Scalar Field

Recall Lagrangian in Eq. (2.2)

$$\partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{2.12}$$

On p. 5 we had shown that Eq. (2.12) is invariant under $\phi \to e^{i\alpha}\phi$. What if now $\alpha = \alpha(x)$, i.e. it depends on spacetime?

$$\begin{aligned}
\partial_\mu (e^{i\alpha(x)}\phi)^* \partial^\mu (e^{i\alpha(x)}\phi) - m^2 (e^{i\alpha(x)}\phi)^* (e^{i\alpha(x)}\phi)& \\
= [\partial_\mu e^{i\alpha(x)} \cdot \phi + e^{i\alpha(x)} \cdot \partial_\mu \phi]^* [\partial^\mu e^{i\alpha(x)} \cdot \phi + e^{i\alpha(x)} \cdot \partial^\mu \phi] - m^2 \phi^* \phi& \\
= [ie^{i\alpha(x)}\partial_\mu \alpha(x) \cdot \phi + e^{i\alpha(x)} \cdot \partial_\mu \phi]^* [ie^{i\alpha(x)}\partial^\mu \alpha(x) \cdot \phi + e^{i\alpha(x)} \cdot \partial^\mu \phi] - m^2 \phi^* \phi& \\
= [-ie^{-i\alpha(x)}\partial_\mu \alpha(x) \cdot \phi^* + e^{-i\alpha(x)} \cdot \partial_\mu \phi^*][ie^{i\alpha(x)}\partial^\mu \alpha(x) \cdot \phi + e^{i\alpha(x)} \cdot \partial^\mu \phi] - m^2 \phi^* \phi& \\
= -ie^{-i\alpha(x)}\partial_\mu \alpha(x) \cdot \phi^* \cdot ie^{i\alpha(x)}\partial^\mu \alpha(x) \cdot \phi& \\
- ie^{-i\alpha(x)}\partial_\mu \alpha(x) \cdot \phi^* \cdot e^{i\alpha(x)} \cdot \partial^\mu \phi& \\
+ e^{-i\alpha(x)} \cdot \partial_\mu \phi^* \cdot ie^{i\alpha(x)}\partial^\mu \alpha(x) \cdot \phi& \\
+ e^{-i\alpha(x)} \cdot \partial_\mu \phi^* \cdot e^{i\alpha(x)} \cdot \partial^\mu \phi& \\
- m^2 \phi^* \phi& \\
= \partial_\mu \phi \cdot \partial^\mu \phi - m^2 \phi^* \phi + \text{non-zero terms}&
\end{aligned}$$

Not invariant under U(1)! The reason why it worked before was that $\partial_\mu[e^{i\alpha}\cdot] = e^{i\alpha}\partial_\mu[\cdot]$. Can we find a derivative operator that commutes with the gauge transformation?

$$D_\mu[e^{i\alpha(x)}\cdot] = e^{i\alpha(x)}D_\mu[\cdot] \tag{2.13}$$

Define

$$D_\mu = \partial_\mu + iA_\mu, \tag{2.14}$$

where the *gauge field* A_μ transforms as

$$A_\mu \to A_\mu - \partial_\mu\alpha \tag{2.15}$$

under the gauge transformation. Now we can try again. Is

$$D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi \tag{2.16}$$

invariant under $\phi \to e^{i\alpha(x)}\phi$? I could repeat the previous calculation, but it is more instructive to take a short-cut and prove Eq. (2.13) instead. The reason is that this will also generalize to the non-Abelian case.

$$\begin{aligned} D_\mu\phi &\to (\partial_\mu + i[A_\mu - \partial_\mu\alpha(x)])[e^{i\alpha(x)}\phi] \\ &= \partial_\mu[e^{i\alpha(x)}\phi] + i[A_\mu - \partial_\mu\alpha(x)][e^{i\alpha(x)}\phi] \\ &= ie^{i\alpha(x)}\partial_\mu\alpha(x)\cdot\phi + e^{i\alpha(x)}\partial_\mu\phi + iA_\mu e^{i\alpha(x)}\phi - i\partial_\mu\alpha(x)e^{i\alpha(x)}\phi \\ &= e^{i\alpha(x)}\partial_\mu\phi + iA_\mu e^{i\alpha(x)}\phi \\ &= e^{i\alpha(x)}[\partial_\mu\phi + iA_\mu]\phi \\ &= e^{i\alpha(x)}D_\mu\phi \end{aligned} \tag{2.17}$$

From this, it directly follows that Eq. (2.16) is invariant:

$$D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi \to e^{-i\alpha(x)}D_\mu\phi^*\cdot e^{i\alpha(x)}D^\mu\phi - m^2e^{-i\alpha(x)}\phi^*\cdot e^{i\alpha(x)}\phi = D_\mu\phi^* D^\mu\phi - m^2$$

Now you can expand Eq. (2.16) to discover the consequences of gauge invariance:

$$D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi = \partial_\mu\phi^*\partial^\mu\phi + iA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + \phi^*\phi A_\mu A^\mu - m^2\phi^*\phi \tag{2.18}$$

Nota bene:

- We call D_μ the *covariant derivative*, because it transforms just like ϕ itself:

$$\phi \to e^{i\alpha(x)}\phi \quad \text{and} \quad D_\mu\phi \to e^{i\alpha(x)}D_\mu\phi \tag{2.19}$$

- Demand symmetry → Generate interactions
- Generated mass for gauge boson (after ϕ acquires a vacuum expectation value)
- Explicit mass term forbidden by gauge symmetry (although otherwise allowed):

$$m^2 A_\mu A^\mu \to m^2(A_\mu - \partial_\mu \alpha)(A_\mu - \partial_\mu \alpha) \neq m^2 A_\mu A^\mu \tag{2.20}$$

- Simplest form of Higgs mechanism
- Vector-scalar-scalar interaction

Homework 2.2 *Define the covariant derivative*

$$D_\mu = \partial_\mu - igA_\mu^a T_R^a \tag{2.21}$$

where g is the gauge coupling and T_R^a are the representation matrices of the Lie algebra elements T^a (the subscript R reminds us that we are working in a given representation). Under a gauge transformation

$$U = e^{ig\alpha^a(x)T_R^a} \tag{2.22}$$

the field ϕ transforms as

$$\phi \to U\phi \tag{2.23}$$

and we define

$$A_\mu \to UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger. \tag{2.24}$$

Note that U is a matrix and depends on the representation of the Lie algebra in which ϕ transforms (choice of T_R^a in Eq. (2.22)). Show that

$$D_\mu\phi \to UD_\mu\phi, \tag{2.25}$$

i.e. $D_\mu\phi$ transforms covariantly. You can proceed in several steps of increasing complexity:

(a) Assume that $a = 1$, $T_R^a = 1$, $g = 1$. This should (up to a minus sign here and there) reproduce our calculation in Eq. (2.17).

(b) Assume now that $a = 1, 2, 3$ and $T_R^a = \sigma_a$ are the Pauli matrices. This case corresponds to a representation which acts on a doublet. In the SM, this corresponds to e.g. $Q \to e^{ig\alpha^a(x)\sigma_a}Q$ where $Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$. Note that u_L and d_L are Weyl spinors, whereas Q is not! Q is a doublet under $\mathrm{SU}(2)_L$.

(c) Now prove the general case. This should be almost the same as the previous proof for the $\mathrm{SU}(2)$ case.

Adding the Gauge Fields

Recall the gauge invariant Lagrangian for a complex scalar field from Eq. (2.16):

$$D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi \tag{2.26}$$

When defining the covariant derivative, we were led to introduce *gauge field* A^a_μ. Since these fields are now present in the theory, we need to introduce kinetic terms for them (note that mass terms are forbidden by gauge invariance, see Eq. (2.20) on the preceding page and Eq. (2.9) on page 6):

$$D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{2.27}$$

Consider first the case of a U(1) gauge field:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.28}$$

It is easy to prove that this term is gauge invariant:

$$\begin{aligned} F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu &\to \partial_\mu (A_\nu - \partial_\nu \alpha(x)) - \partial_\nu (A_\mu - \partial_\mu \alpha(x)) \\ &= \partial_\mu A_\nu - \partial_\mu \partial_\nu \alpha(x) - \partial_\nu A_\mu - \partial_\nu \partial_\mu \alpha(x) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \end{aligned} \tag{2.29}$$

For the non-Abelian case (e.g. SU(2)), the situation is more complicated, and we need to amend the definition of $F_{\mu\nu}$ to make the product $F_{\mu\nu} F^{\mu\nu}$ gauge invariant. Here is a short overview of the differences between the abelian and non-abelian case:

Abelian	Non-Abelian: component notation	Non-Abelian: vector notation
$U = e^{i\alpha(x)}$	$U = e^{ig\alpha^a(x) T^a_R}$	$U = e^{ig\alpha^a(x) T^a_R}$
$\phi \to U\phi$	$\mathbf{\Phi}^i \to U^i{}_k \mathbf{\Phi}^k$	$\mathbf{\Phi} \to U\mathbf{\Phi}$
A_μ	$A^a_\mu T^a_R$	$\boldsymbol{A}_\mu$
$A_\mu \to A_\mu - \partial_\mu \alpha$	$A^a_\mu T^a \to U A^a_\mu T^a U^\dagger - \frac{i}{g}(\partial_\mu U) U^\dagger$	$\boldsymbol{A}_\mu \to U \boldsymbol{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U) U^\dagger$
$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$	$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$	$\boldsymbol{F}_{\mu\nu} := \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu - ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]$
$F_{\mu\nu} \to F_{\mu\nu}$		$\boldsymbol{F}_{\mu\nu} \to U \boldsymbol{F}_{\mu\nu} U^\dagger$
$F_{\mu\nu}$ invariant	$F^a_{\mu\nu} F^{a\mu\nu}$ invariant	$\mathrm{Tr}(\boldsymbol{F}_{\mu\nu} \boldsymbol{F}^{\mu\nu})$ invariant

Homework 2.3 *Prove that*

$$\frac{1}{2}\mathrm{Tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) = \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}. \tag{2.30}$$

Hint: $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.

Homework 2.4 *Prove that if we define the gauge fields to transform as*

$$\boldsymbol{A}_\mu \to U\boldsymbol{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger, \tag{2.31}$$

then the field strenght tensor will transform as

$$\boldsymbol{F}_{\mu\nu} \to U\boldsymbol{F}_{\mu\nu}U^\dagger. \tag{2.32}$$

Homework 2.5 *Prove that*

$$\mathrm{Tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu}) \tag{2.33}$$

is invariant.

3 Supersymmetry

3.1 Weyl Spinors (in SUSY Notation)

Remember the Lagrangian for a massive fermion:

$$\mathcal{L} = i\psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L) \tag{3.1}$$

Let us single out the term $\psi_R^\dagger \psi_L$. First, we note that it is Lorentz invariant. Second, we can rewrite it as the product of two left-handed Weyl spinors, since $\psi_R^c := -i\sigma_2 \psi_R^*$ transforms as a left-handed spinor:

$$\psi_R^c = -i\sigma_2 \psi_R^* \leftrightarrow +i\sigma_2 \psi_R^c = \psi_R^* \leftrightarrow (\psi_R^c)^T(-i\sigma_2) = \psi_R^\dagger \quad \rightsquigarrow \quad \psi_R^\dagger \psi_L = (\psi_R^c)^T(-i\sigma_2)\psi_L \tag{3.2}$$

Now introduce new names $\xi := \psi_R^c$, $\chi := \psi_L$ for the 2 left-handed Weyl spinors, use component notation and rename $\epsilon_{\alpha\beta} := -(i\sigma_2)_{\alpha\beta}$ which now plays the role of a "spinor metric":

$$\psi_R^\dagger \psi_L = (\psi_R^c)^T(-i\sigma_2)\psi_L = \xi^T(-i\sigma_2)\chi = \xi^\alpha \epsilon_{\alpha\beta}\chi^\beta \tag{3.3}$$

Note that $\epsilon_{\alpha\beta}$ is indeed our old friend the totally anti-symmetric 2-index tensor:

$$\epsilon_{12} = -\epsilon_{21} = -1, \quad \epsilon_{11} = -\epsilon_{22} = 0 \tag{3.4}$$

Consequently, we have to define

$$\epsilon^{12} = -\epsilon^{21} = +1, \quad \epsilon^{11} = -\epsilon^{22} = 0 \tag{3.5}$$

such that $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta$.

Note that we agree with the conventions of Martin [3], but this need not have been the case. From the derivation you can see that the sign of ϵ_{12} *is a convention.*

An immediate consequence of Eq. (3.4) and Eq. (3.5) is that

$$\xi\chi \stackrel{?}{=} \xi^\alpha \chi_\alpha = \epsilon^{\alpha\beta}\xi_\beta \epsilon_{\alpha\gamma}\chi^\gamma = \epsilon^{\alpha\beta}\epsilon_{\alpha\gamma}\xi_\beta\chi^\gamma = -\epsilon^{\beta\alpha}\epsilon_{\alpha\gamma}\xi_\beta\chi^\gamma = -\delta_\gamma^\beta \xi_\beta \chi^\gamma = -\xi_\beta\chi^\beta \stackrel{?}{=} \xi\chi. \tag{3.6}$$

We see that we need another convention in order to unambigously define what we mean by $\xi\chi$ with indices suppressed. We agree to sum undotted indices always "downwards", and dotted indices always "upwards".

Note added: We chose to define

$$\xi_\alpha = \epsilon_{\alpha\beta}\xi^\beta, \qquad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta \tag{3.7}$$

as Martin [3] did, and this is the "technical" reason why we are getting a minus sign in Eq. (3.6). However, this is again a convention, and e.g. Kalka and Soff [1] defines

$$\xi_\alpha = \xi^\beta\epsilon_{\beta\alpha}, \qquad \xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta, \tag{3.8}$$

which is in some sense more natural, since it mimics the transformation property of the Weyl spinors ψ_L and ψ_R. This leads to the same result as in Eq. (3.6), as one may easily check. Just note that in the notation of Kalka and Soff [1], $\epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = -\delta_\alpha^\beta$ *in contrast to* Martin [3] (see line following Eq. (3.5)). I inserted this short note to answer Tim's question. □

An important point is that the components of ξ and χ are so-called a-numbers (anti-commuting numbers), i.e. e.g. $\xi_\alpha\chi_\beta = -\chi_\beta\xi_\alpha$. This is of course so because later in second-quantization we need anti-commutation relations (beyond the scope of this lecture).

Following the literature, we introduce the (somewhat redundant) notation and denote the indices of conjugate spinors with dots, i.e.

$$\psi \sim \xi^\alpha \quad \rightsquigarrow \quad \psi^\dagger \sim \xi^{\dagger\dot\alpha}. \tag{3.9}$$

Sometimes $\xi^\dagger$ is called a right-handed Weyl spinor, but this is misleading, since without referring to the spinor metric $\epsilon_{\alpha\beta}$, this does not make sense (cmp. Eq. (3.1) and Eq. (3.10) below).

Now we can rewrite Eq. (3.1) as:

$$\mathcal{L} = i\xi^\dagger\overline{\sigma}^\mu\partial_\mu\xi + i\chi^\dagger\overline{\sigma}^\mu\partial_\mu\chi - m(\xi\chi + \xi^\dagger\chi^\dagger) \tag{3.10}$$

3.2 The Chiral Supermultiplet

Before we start, I should mention that our notation here slightly differs from Martin [3], since among other things our metric is "mostly minus", i.e. $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$, whereas his $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$ is "mostly plus". You should keep this in mind, if you want to compare with Martin [3].

This brings me to an important point that I had also mentioned before, namely that in SUSY the choice of *notation and conventions* is particularly important. There are no two authors that use the same notation and conventions (slight exaggeration), so when you develop your arsenal of formulae, you need to fix your notation and stick to it. Otherwise, you will unevitably end up with errors and mistakes in your calculations and look for minus signs for hours and hours.

3.2.1 On-shell Formulation

SUSY associates a fermionic partner with every boson and vice versa, so let us start with the simplest case. We are looking for a transformation such that

$$S = \int d^4x \left(\partial_\mu \phi^* \partial^\mu \phi - i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi\right) =: \int d^4x \left(\mathcal{L}_\phi + \mathcal{L}_\psi\right) \tag{3.11}$$

is invariant.

The desired transformation must turn a boson into a fermion. We make the ansatz

$$\phi \to \phi + \delta\phi, \quad \text{where} \quad \delta\phi = \epsilon\psi \quad \text{and} \quad \delta\phi^* = \epsilon^\dagger \psi^\dagger. \tag{3.12}$$

Do not worry that this might not be the most general SUSY transformation. We are looking for one realization of the SUSY algebra, just like the Pauli matrices are *a* (not the!) realization of the SU(2) algebra. Later in this chapter we will show that Eq. (3.12) and the corresponding transformation of the fermions indeed satisfy the SUSY algebra. Also note that Eq. (3.12) is linear in ϕ, so we are looking for a linear realization of the SUSY algebra. One last remark, namely note that ϵ does not depend on x, i.e. $\epsilon \neq \epsilon(x)$. In analogy to before, we call this a *global* SUSY transformation. If $\epsilon = \epsilon(x)$, we have a *local* SUSY transformation, and the resulting theory is called supergravity.

So, how does $\mathcal{L}_\phi$ transform under Eq. (3.12)?

$$\begin{aligned}
\delta\mathcal{L}_\phi &= \delta\left(\partial_\mu\phi^*\partial^\mu\phi\right) = \delta\left(\partial_\mu\phi^*\right)\partial^\mu\phi + \partial_\mu\phi^*\delta\left(\partial^\mu\phi\right) = \partial_\mu\left(\delta\phi^*\right)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\left(\delta\phi\right) \\
&= \partial_\mu\left(\delta\phi^*\right)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\left(\delta\phi\right) \\
&= \partial_\mu\left(\epsilon^\dagger\psi^\dagger\right)\partial^\mu\phi + \partial_\mu\phi^*\partial^\mu\left(\epsilon\psi\right) \qquad \text{see Eq. (3.12)} \\
&= \epsilon^\dagger\partial_\mu\psi^\dagger\partial^\mu\phi + \partial_\mu\phi^*\epsilon\partial^\mu\psi \qquad \epsilon \neq \epsilon(x)
\end{aligned} \tag{3.13}$$

Now consider the transformation properties of $\mathcal{L}_\psi$. How must ψ transform such that the resulting transformation of $\mathcal{L}_\psi$ cancels the terms in Eq. (3.13)? One subtlety: We need the action S in Eq. (3.11) to be invariant; the Lagrangian $\mathcal{L}_\phi + \mathcal{L}_\psi$ need not necessarily be invariant; we can allow for a *total derivative.*

This time, we do not have any choice. We look for a transformation $\delta\psi$ that should contain ϵ, ϕ, and one derivative ∂_μ. Otherwise, there is no chance that it will cancel the terms in Eq. (3.13). But we also need to contract the Lorentz index μ of the derivative ∂_μ with something, otherwise the term will not be Lorentz invariant. The only option is to introduce a σ^μ. We make the following ansatz:

$$\psi \to \psi + \delta\psi, \quad \text{where} \quad \delta\psi_\alpha = +i\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dagger\dot\alpha}\partial_\mu\phi \quad \text{and} \quad \delta\psi^\dagger_{\dot\alpha} = -i\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi^*. \tag{3.14}$$

Note that whether we should take an ϵ or $\epsilon^\dagger$ is dictated by the fact that for $\delta\psi$ we need an α-index, and for $\delta\psi^\dagger$ we need an $\dot\alpha$-index:

$$\sigma^\mu_{\alpha\dot\alpha}(\epsilon^\dagger)^{\dot\alpha} = (\text{something})_\alpha, \qquad \epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha} = (\text{something})_{\dot\alpha} \tag{3.15}$$

Also, we cannot take a $(\overline{\sigma}^\mu)^{\dot\alpha\alpha}$, since its indices are up, and we need something to match the down-index of $\delta\psi_\alpha$. The bottom line is that Eq. (3.14) is the only sensible choice.

Homework 3.1 *Prove the following identities:*

$$\xi\chi = \chi\xi \quad \textit{and} \quad \xi^\dagger\chi^\dagger = \chi^\dagger\xi^\dagger \tag{3.16}$$

$$(\sigma^\mu\overline{\sigma}^\nu + \sigma^\nu\overline{\sigma}^\mu)_\alpha{}^\beta = 2\eta^{\mu\nu}\delta_\alpha^\beta \tag{3.17}$$

$$(\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu)^{\dot\beta}{}_{\dot\alpha} = 2\eta^{\mu\nu}\delta_{\dot\alpha}^{\dot\beta} \tag{3.18}$$

How does $\mathcal{L}_\psi$ transform under Eq. (3.14)?

$$\begin{aligned}
-\delta\mathcal{L}_\psi &= \delta\left(i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi\right)\\
&= \delta\left(i\psi^\dagger_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\alpha}\partial_\mu\psi_\alpha\right)\\
&= i\delta\psi^\dagger_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\alpha}\partial_\mu\psi_\alpha + i\psi^\dagger_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\alpha}\partial_\mu(\delta\psi_\alpha)\\
&= i(-i\epsilon^\beta\sigma^\nu_{\beta\dot\alpha}\partial_\nu\phi^*)\overline{\sigma}^{\mu\dot\alpha\alpha}\partial_\mu\psi_\alpha + i\psi^\dagger_{\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\alpha}\partial_\mu(+i\sigma^\nu_{\alpha\dot\beta}\epsilon^{\dagger\dot\beta}\partial_\nu\phi)\\
&= \epsilon\sigma^\nu\overline{\sigma}^\mu\partial_\mu\psi\partial_\nu\phi^* - \psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\partial_\mu\partial_\nu\phi \qquad \Big| f'g = (fg)' - f'g\\
&= [\partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\mu\partial_\nu\phi^*] - \psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\partial_\mu\partial_\nu\phi \qquad \Big| A = \frac{1}{2}(A+A)\\
&= \left[\partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \frac{1}{2}\left(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\mu\partial_\nu\phi^* + \epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\mu\partial_\nu\phi^*\right)\right] - \frac{1}{2}\left(\psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\partial_\mu\partial_\nu\phi + \psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\partial_\mu\partial_\nu\phi\right)\\
&= \left[\partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \frac{1}{2}\left(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\mu\partial_\nu\phi^* + \epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\partial_\mu\phi^*\right)\right] - \frac{1}{2}\left(\psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\partial_\mu\partial_\nu\phi + \psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\partial_\nu\partial_\mu\phi\right)\\
&= \left[\partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \frac{1}{2}\left(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\mu\partial_\nu\phi^* + \epsilon\sigma^\mu\overline{\sigma}^\nu\psi\partial_\mu\partial_\nu\phi^*\right)\right] - \frac{1}{2}\left(\psi^\dagger\overline{\sigma}^\mu\sigma^\nu\epsilon^\dagger\partial_\mu\partial_\nu\phi + \psi^\dagger\overline{\sigma}^\nu\sigma^\mu\epsilon^\dagger\partial_\mu\partial_\nu\phi\right)\\
&= \left[\partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \frac{1}{2}\epsilon\left(\sigma^\nu\overline{\sigma}^\mu + \sigma^\mu\overline{\sigma}^\nu\right)\psi\partial_\mu\partial_\nu\phi^*\right] - \frac{1}{2}\psi^\dagger\left(\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu\right)\epsilon^\dagger\partial_\mu\partial_\nu\phi \qquad \text{Indices!}\\
&= \left[\partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \frac{1}{2}\epsilon^\alpha\left(\sigma^\nu\overline{\sigma}^\mu + \sigma^\mu\overline{\sigma}^\nu\right)_\alpha{}^\beta\psi_\beta\partial_\mu\partial_\nu\phi^*\right] - \frac{1}{2}\psi^\dagger_{\dot\beta}\left(\overline{\sigma}^\mu\sigma^\nu + \overline{\sigma}^\nu\sigma^\mu\right)^{\dot\beta}{}_{\dot\alpha}\epsilon^{\dagger\dot\alpha}\partial_\mu\partial_\nu\phi \qquad \Big|\text{Eq. (3.17)}\\
&= \left[\partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \epsilon^\alpha\eta^{\mu\nu}\delta_\alpha^\beta\psi_\beta\partial_\mu\partial_\nu\phi^*\right] - \psi^\dagger_{\dot\beta}\eta^{\mu\nu}\delta_{\dot\alpha}^{\dot\beta}\epsilon^{\dagger\dot\alpha}\partial_\mu\partial_\nu\phi\\
&= [\partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \epsilon^\alpha\psi_\alpha\partial_\mu\partial^\mu\phi^*] - \psi^\dagger_{\dot\alpha}\epsilon^{\dagger\dot\alpha}\partial_\mu\partial^\mu\phi\\
&= \partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \epsilon\psi\partial_\mu\partial^\mu\phi^* - \psi^\dagger\epsilon^\dagger\partial_\mu\partial^\mu\phi \qquad \Big|\text{Eq. (3.16)}\\
&= \partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - \epsilon\psi\partial_\mu\partial^\mu\phi^* - \epsilon^\dagger\psi^\dagger\partial_\mu\partial^\mu\phi\\
&= \partial_\mu(\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^*) - [\partial_\mu(\epsilon\psi\partial^\mu\phi^*) - \epsilon\partial_\mu\psi\partial^\mu\phi^*] - \left[\partial_\mu(\epsilon^\dagger\psi^\dagger\partial^\mu\phi) - \epsilon^\dagger\partial_\mu\psi^\dagger\partial^\mu\phi\right] \qquad \Big| f'g = (fg)' - f'g\\
&= \epsilon\partial_\mu\psi\partial^\mu\phi^* + \epsilon^\dagger\partial_\mu\psi^\dagger\partial^\mu\phi + \partial_\mu\left[\epsilon\sigma^\nu\overline{\sigma}^\mu\psi\partial_\nu\phi^* - \epsilon\psi\partial^\mu\phi^* - \epsilon^\dagger\psi^\dagger\partial^\mu\phi\right] &(3.19)\\
&= -\delta\mathcal{L}_\psi &(3.20)
\end{aligned}$$

We can now easily show that Eq. (3.11) is invariant under SUSY transformations:

$$\begin{aligned}
\delta S &= \int d^4x\ (\delta\mathcal{L}_\phi + \delta\mathcal{L}_\psi) \\
&= \int d^4x \Big\{ \big[\epsilon^\dagger \partial_\mu \psi^\dagger \partial^\mu \phi + \partial_\mu \phi^* \epsilon \partial^\mu \psi\big] \\
&\quad - \big[\epsilon \partial_\mu \psi \partial^\mu \phi^* + \epsilon^\dagger \partial_\mu \psi^\dagger \partial^\mu \phi + \partial_\mu \big(\epsilon \sigma^\nu \overline{\sigma}^\mu \psi \partial_\nu \phi^* - \epsilon \psi \partial^\mu \phi^* - \epsilon^\dagger \psi^\dagger \partial^\mu \phi\big)\big] \Big\} \\
&= -\int d^4x\, \partial_\mu \big(\epsilon \sigma^\nu \overline{\sigma}^\mu \psi \partial_\nu \phi^* - \epsilon \psi \partial^\mu \phi^* - \epsilon^\dagger \psi^\dagger \partial^\mu \phi\big)
\end{aligned} \tag{3.21}$$

This is a total derivative and (by the usual arguments) does not contribute to the *action* S given in Eq. (3.11).

3.2.2 The SUSY Algebra

What we have shown so far is that the transformation of a boson into a fermion and vice versa (as given in Eq. (3.12) and Eq. (3.14)) leaves the action in Eq. (3.11) invariant. We still do not know whether these transformations consistently define an algebra. For that, we need to show that there is a set of generators which transform into each other and do not lead to results outside the algebra. Strictly speaking, the SUSY generators are part of what we call a "graded Lie algebra", but for the time being, we need not indulge in such details. Let us start by exploring what the commutator of two SUSY generators looks like. Note that in the context of an algebra, only the commutator is well-defined. In general, it does not make sense to talk about the product of two elements of an algebra (whereas in a group, it is of course fine).

Let δ_{ϵ_1} and δ_{ϵ_2} denote 2 SUSY transformations. Note that we have to chose 2 different values ϵ_1 and ϵ_2 for the infinitesimal parameter of the SUSY transformation, otherwise it will be the same transformation and trivially commute. Let us first apply it to a bosonic field ϕ:

$$\begin{aligned}
(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})\phi &= \delta_{\epsilon_1}(\delta_{\epsilon_2}\phi) - \delta_{\epsilon_2}(\delta_{\epsilon_1}\phi) \qquad \Big| \text{Eq. (3.12)} \\
&= \delta_{\epsilon_1}(\epsilon_2\psi) - \delta_{\epsilon_2}(\epsilon_1\psi) \\
&= \epsilon_2\delta_{\epsilon_1}\psi - \epsilon_1\delta_{\epsilon_2}\psi \qquad \Big| \text{Eq. (3.14)} \\
&= \epsilon_2^\alpha(+i\sigma^\mu_{\alpha\dot{\alpha}}\epsilon_1^{\dagger\dot{\alpha}}\partial_\mu\phi) - \epsilon_1^\alpha(+i\sigma^\mu_{\alpha\dot{\alpha}}\epsilon_2^{\dagger\dot{\alpha}}\partial_\mu\phi) \\
&= \epsilon_2^\alpha(+i\sigma^\mu_{\alpha\dot{\alpha}}\epsilon_1^{\dagger\dot{\alpha}}\partial_\mu\phi) - \epsilon_1^\alpha(+i\sigma^\mu_{\alpha\dot{\alpha}}\epsilon_2^{\dagger\dot{\alpha}}\partial_\mu\phi) \\
&= i(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger)\partial_\mu\phi \\
&= \epsilon_3\ i\partial_\mu\phi \qquad \text{with } \epsilon_3 := \epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger
\end{aligned} \tag{3.22}$$

From quantum field theory we know that $i\partial_\mu$ corresponds to the momentum operator

P_μ, so we have just proven that the commutator of two SUSY transformations is a translation (in space and time).

Homework 3.2 *Prove the following identities:*

$$\chi_\alpha(\xi\eta) = -\xi_\alpha(\eta\chi) - \eta_\alpha(\chi\xi) \quad \textit{(Fierz identity)} \tag{3.23}$$

$$\xi^\dagger\overline{\sigma}^\mu\chi = -\chi\sigma^\mu\xi^\dagger = (\chi^\dagger\overline{\sigma}^\mu\xi)^* = -(\xi\sigma^\mu\chi^\dagger)^* \tag{3.24}$$

Here is a proof of the Fierz identity:

$$\begin{aligned}
\epsilon_{\alpha\beta}(\chi\eta) &= \epsilon_{\alpha\beta}\chi^\gamma\eta_\gamma = \epsilon_{\alpha\beta}\epsilon^{\gamma\delta}\chi_\delta\eta_\gamma = \epsilon_{\alpha\beta}\epsilon^{\gamma\delta}\chi_\delta\eta_\gamma \\
&= (\delta^\delta_\alpha\delta^\gamma_\beta - \delta^\gamma_\alpha\delta^\delta_\beta)\chi_\delta\eta_\gamma = \chi_\alpha\eta_\beta - \chi_\beta\eta_\alpha
\end{aligned}$$

$$\begin{aligned}
\chi_\alpha(\xi\eta) &= \chi_\alpha\xi^\beta\eta_\beta = -\xi^\beta\chi_\alpha\eta_\beta = -\epsilon^{\beta\gamma}\xi_\gamma\chi_\alpha\eta_\beta \\
&= -\epsilon^{\beta\gamma}\xi_\gamma(\chi_\beta\eta_\alpha + \epsilon_{\alpha\beta}(\chi\eta)) \\
&= -\epsilon^{\beta\gamma}\xi_\gamma\chi_\beta\eta_\alpha - \epsilon^{\beta\gamma}\epsilon_{\alpha\beta}\xi_\gamma(\chi\eta) \\
&= -\xi^\beta\chi_\beta\eta_\alpha - \epsilon^{\gamma\beta}\epsilon_{\beta\alpha}\xi_\gamma(\chi\eta) \\
&= -(\xi\chi)\eta_\alpha - \delta^\gamma_\alpha\xi_\gamma(\chi\eta) \\
&= -(-1)^2\eta_\alpha(\xi\chi) - \xi_\alpha(\chi\eta) \qquad \text{Q.E.D.}
\end{aligned}$$

Let us now repeat the same calculation for a fermion field:

$$\begin{aligned}
(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})\psi_\alpha &= \delta_{\epsilon_1}(\delta_{\epsilon_2}\psi_\alpha) - \delta_{\epsilon_2}(\delta_{\epsilon_1}\psi_\alpha) && \Big|\text{Eq. (3.14)} \\
&= \delta_{\epsilon_1}\left(+i\sigma^\mu_{\alpha\dot\alpha}\epsilon_2^{\dagger\dot\alpha}\partial_\mu\phi\right) - \delta_{\epsilon_2}\left(+i\sigma^\mu_{\alpha\dot\alpha}\epsilon_1^{\dagger\dot\alpha}\partial_\mu\phi\right) \\
&= \left(+i\sigma^\mu_{\alpha\dot\alpha}\epsilon_2^{\dagger\dot\alpha}\partial_\mu\delta_{\epsilon_1}\phi\right) - \left(+i\sigma^\mu_{\alpha\dot\alpha}\epsilon_1^{\dagger\dot\alpha}\partial_\mu\delta_{\epsilon_2}\phi\right) && \Big|\text{Eq. (3.12)} \\
&= \left(+i\sigma^\mu_{\alpha\dot\alpha}\epsilon_2^{\dagger\dot\alpha}\partial_\mu(\epsilon_1\psi)\right) - \left(+i\sigma^\mu_{\alpha\dot\alpha}\epsilon_1^{\dagger\dot\alpha}\partial_\mu(\epsilon_2\psi)\right) \\
&= \left(+i\sigma^\mu_{\alpha\dot\alpha}\epsilon_2^{\dagger\dot\alpha}\epsilon_1\partial_\mu\psi\right) - \left(+i\sigma^\mu_{\alpha\dot\alpha}\epsilon_1^{\dagger\dot\alpha}\epsilon_2\partial_\mu\psi\right) \\
&= i\sigma^\mu_{\alpha\dot\alpha}\epsilon_2^{\dagger\dot\alpha}\epsilon_1^\beta\partial_\mu\psi_\beta - i\sigma^\mu_{\alpha\dot\alpha}\epsilon_1^{\dagger\dot\alpha}\epsilon_2^\beta\partial_\mu\psi_\beta && \Big|\text{Just to make the index structure clear!} \\
&= i(\sigma^\mu\epsilon_2^\dagger)_\alpha(\epsilon_1\partial_\mu\psi) - i(\sigma^\mu\epsilon_1^\dagger)_\alpha(\epsilon_2\partial_\mu\psi) && \Big|\text{Eq. (3.23): } \chi_\alpha = (\sigma^\mu\epsilon_2^\dagger)_\alpha, \xi = \epsilon_1, \eta = \partial_\mu\psi \\
&= i\left[-\epsilon_{1\alpha}(\partial_\mu\psi\sigma^\mu\epsilon_2^\dagger) - \partial_\mu\psi_\alpha(\sigma^\mu\epsilon_2^\dagger\epsilon_1)\right] - i\left[-\epsilon_{2\alpha}(\partial_\mu\psi\sigma^\mu\epsilon_1^\dagger) - \partial_\mu\psi_\alpha(\sigma^\mu\epsilon_1^\dagger\epsilon_2)\right] && \Big|\text{Collect } \psi_\alpha \\
&= i\left(\sigma^\mu\epsilon_1^\dagger\epsilon_2 - \sigma^\mu\epsilon_2^\dagger\epsilon_1\right)\partial_\mu\psi_\alpha - i\epsilon_{1\alpha}(\partial_\mu\psi\sigma^\mu\epsilon_2^\dagger) + i\epsilon_{2\alpha}(\partial_\mu\psi\sigma^\mu\epsilon_1^\dagger) && \Big|\text{Eq. (3.16)} \\
&= i\left(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger\right)\partial_\mu\psi_\alpha - i\epsilon_{1\alpha}(\partial_\mu\psi\sigma^\mu\epsilon_2^\dagger) + i\epsilon_{2\alpha}(\partial_\mu\psi\sigma^\mu\epsilon_1^\dagger) && \Big|\text{Eq. (3.24)} \\
&= i\left(\epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger\right)\partial_\mu\psi_\alpha + i\epsilon_{1\alpha}(\epsilon_2^\dagger\overline{\sigma}^\mu\partial_\mu\psi) - i\epsilon_{2\alpha}(\epsilon_1^\dagger\overline{\sigma}^\mu\partial_\mu\psi) && (3.25)
\end{aligned}$$

Note that the last two terms vanish because of the (Weyl) equation of motion: $\overline{\sigma}^\mu \partial_\mu \psi = 0$. Then Eq. (3.25) reduces to the generator of spacetime translations P_μ with the same infinitesimal parameter $\epsilon_3 := \epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger$ as in Eq. (3.22). As a result, we have shown that also for fermions, the commutator of two SUSY transformations is the 4-momentum operator.

3.2.3 Off-shell Formulation

In Section 3.2.2 we have shown that the SUSY algebra closes, if we use the *classical* equations of motion. We would like SUSY to be a symmetry of the Lagrangian itself. This will then guarantee that SUSY will also be respected at the quantum level. To this end, we introduce an *auxiliary* complex scalar field F with the following Lagrangian density:

$$\mathcal{L}_F = F^* F \tag{3.26}$$

Its equation of motion is

$$\partial_\mu \frac{\partial \mathcal{L}_F}{\partial(\partial_\mu F)} - \frac{\mathcal{L}_F}{\partial F} = 0 \qquad \rightsquigarrow \qquad F^* = F = 0 \tag{3.27}$$

Note that the field F has no propagating degrees of freedom. It is only a *book-keeping device* and has no physical significance. We simply use it as a trick to render the Lagrangian supersymmetric even if we are *off-shell*, i.e. not using the equations of motion.

We now have to figure out the transformation properties of F. Since we introduced it to cancel the unwanted terms in Eq. (3.25) (last 2 terms), it makes sense to make the following ansatz:

$$F \to F + \delta F, \quad \text{where} \quad \delta F = +i\epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi \quad \text{and} \quad \delta F^* = -i\partial_\mu \psi^\dagger \overline{\sigma}^\mu \epsilon \tag{3.28}$$

To see whether this ansatz works one has to repeat the calculation in Eq. (3.25) (the answer is yes). However, since we modified the Lagrangian,

$$S = \int d^4x \, (\partial_\mu \phi^* \partial^\mu \phi + i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi + F^* F) =: \int d^4x \, (\mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_F), \tag{3.29}$$

we cannot expect the new Lagrangian to be invariant under the SUSY transformations, and indeed, it is not. We can easily fix that by changing the transformation properties of ψ:

$$\psi \to \psi + \delta\psi, \quad \text{where} \quad \delta\psi_\alpha = +i\sigma^\mu_{\alpha\dot{\alpha}} \epsilon^{\dagger\dot{\alpha}} \partial_\mu \phi + \epsilon_\alpha F \quad \text{and} \quad \delta\psi^\dagger_{\dot{\alpha}} = -i\epsilon^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \partial_\mu \phi^* + \epsilon^\dagger_{\dot{\alpha}} F^* \tag{3.30}$$

Note that this is plausible, since F is a new *scalar degree of freedom* in the theory, and if we are looking for a transformation that turns a fermion into a boson, we should allow it to appear on the right-hand side of the Eq. (3.30). Out of the same reason, the transformation of ϕ does not change (it changes to a *fermion*).

This field F is simply called "the F-term". It plays an important role in discussing the phenomenology of SUSY (scalar potential, spontaneous SUSY breaking, interactions in the superpotential), so remember its name and where it came from.

Due to time constraints, I will not prove the following important statements which I leave as homework. The calculations are completely analogous to the previous ones, so you should encounter no big problems.

Homework 3.3 *Show that the Lagrangian for the Wess-Zumino model given in Eq. (3.29) on the preceding page is invariant under the SUSY transformations given by Eq. (3.12) on page 14, Eq. (3.30) on the preceding page, and Eq. (3.28) on the previous page.*

Homework 3.4 *Show that the commutator of two SUSY transformations is a spacetime translation, i.e.*

$$(\delta_{\epsilon_1}\delta_{\epsilon_2} - \delta_{\epsilon_2}\delta_{\epsilon_1})X = \epsilon_3\, i\partial_\mu X, \quad \textit{where} \quad X = \phi, \psi_\alpha, F \quad \textit{and} \quad \epsilon_2\sigma^\mu\epsilon_1^\dagger - \epsilon_1\sigma^\mu\epsilon_2^\dagger. \tag{3.31}$$

3.2.4 Adding Interactions

So far we have been working with the *free* Lagrangian

$$\mathcal{L}_{\text{free}} = \mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_F = \partial_\mu\phi^*\partial^\mu\phi - i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + F^*F \tag{3.32}$$

and proved that it is invariant under SUSY transformations. Now we want to add interactions. For ease of reference, we summarize all SUSY transformations in one place:

$$\boxed{\begin{aligned} \delta\phi &= \epsilon\psi & \delta\phi^* &= \epsilon^\dagger\psi^\dagger \\ \delta\psi_\alpha &= +i\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dagger\dot\alpha}\partial_\mu\phi + \epsilon_\alpha F & \delta\psi^\dagger_{\dot\alpha} &= -i\epsilon^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu\phi^* + \epsilon^\dagger_{\dot\alpha}F^* \\ \delta F &= +i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi & \delta F^* &= -i\partial_\mu\psi^\dagger\overline{\sigma}^\mu\epsilon \end{aligned}} \tag{3.33}$$

For the most general interaction Lagrangian $\mathcal{L}_{\text{int}}$, we make the following ansatz:

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}W^{ij}(\phi,\phi^*)\psi_i\psi_j + W^i(\phi,\phi^*)F_i + x^{ij}(\phi,\phi^*)F_iF_j - U(\phi,\phi^*) + \text{c.c.} \tag{3.34}$$

Some remarks are in order:

- i, j are indices that count the Weyl spinors in the theory (e.g. different quarks).
- We have arranged the terms in powers of ψ, i.e. W^{ij}, W^i, x^{ij} and U are polynomials in ϕ and ϕ^*.

- We need not consider terms like F^*F, since they are already part of the free Lagrangian, cf. Eq. (3.32).
- The sum of mass dimensions of each term must be 4. Remember: Scalar=1, Fermion=3/2, F-term=2.
- From this we immediately know that W^{ij}, W^i, x^{ij} and U contain 1, 2, 0, and 4 power(s) of ϕ and/or ϕ^*, respectively.
- A priori there is no connection between W^{ij} and W^i. We have given them similar names, since it will turn out later that they indeed are connected. For the following discussion, we must assume that they are independent.

We will now step-by-step constrain the form of the interaction Lagrangian. For this, we proceed in several steps.

$U(\phi, \phi^*)$ must be identically zero

The term $U(\phi, \phi^*)$ only contains scalar fields, so its SUSY transformation will be (see Eq. (3.33))

$$\delta_\epsilon U \sim \epsilon\psi\widetilde{U}(\phi, \phi^*) \quad \text{or} \quad \delta_\epsilon U \sim \epsilon^\dagger\psi^\dagger\widetilde{U}(\phi, \phi^*), \tag{3.35}$$

where $\widetilde{U}$ is some other function of ϕ and/or ϕ^*. If the Lagrangian in Eq. (3.34) is to be supersymmetric, this contribution needs to be cancelled by one of the other terms in $\mathcal{L}_{\text{int}}$. Note that $\mathcal{L}_{\text{free}}$ in Eq. (3.32) is invariant *by itself* and therefore cannot produce a term that cancels $\delta_\epsilon U$.

Let is go through the terms in $\mathcal{L}_{\text{int}}$ one by one.

The variation of $W^{ij}\psi_i\psi_j$ cannot cancel $\delta_\epsilon U$, since

$$\delta_\epsilon \left(W^{ij}\psi_i\psi_j\right) \sim \delta_\epsilon W^{ij}\psi_i\psi_j + W^{ij}\delta_\epsilon\psi_i\psi_j + W^{ij}\psi_i\delta_\epsilon\psi_j \tag{3.36}$$

will either contain 2 ψ's (first term in Eq. (3.40)) or 1 derivative (second and third term in Eq. (3.40)), and $\delta_\epsilon U$ is not of this form.

The variation of W^iF_i cannot cancel $\delta_\epsilon U$, since

$$\delta_\epsilon \left(W^iF_i\right) \sim \delta_\epsilon W^iF_i + W^i\delta_\epsilon F_i \tag{3.37}$$

will either contain an F-term (first term in Eq. (3.41)) or a derivative (second term in Eq. (3.41)), and $\delta_\epsilon U$ is not of this form.

The variation of $x^{ij}F_iF_j$ cannot cancel $\delta_\epsilon U$, since

$$\delta_\epsilon \left(x^{ij}F_iF_j\right) \sim \delta_\epsilon x^{ij}F_iF_j + x^{ij}\delta_\epsilon F_iF_j + x^{ij}F_i\delta_\epsilon F_j \tag{3.38}$$

will contain at least one F-term, and $\delta_\epsilon U$ is not of this form.

Hence we conclude that $U(\phi, \phi^*) = 0$.

x^{ij} must be identically zero

Above we had already concluded that x^{ij} is a polynomial of degree zero in ϕ and ϕ^*, i.e. it does not contain these fields. Its SUSY transformation is

$$\delta_\epsilon \left(x^{ij} F_i F_j\right) \sim x^{ij} \delta_\epsilon F_i F_j + x^{ij} F_i \delta_\epsilon F_j \sim x^{ij} F\, \epsilon^\dagger \overline{\sigma}^\mu \partial_\mu \psi. \tag{3.39}$$

The variation of $W^{ij}\psi_i\psi_j$ cannot cancel Eq. (3.39), since in

$$\delta_\epsilon \left(W^{ij}\psi_i\psi_j\right) \sim \delta_\epsilon W^{ij}\psi_i\psi_j + W^{ij}\delta_\epsilon\psi_i\psi_j + W^{ij}\psi_i\delta_\epsilon\psi_j \tag{3.40}$$

the first term has 2 ψ's, and the second and third terms will either contain no F-term or no derivative.

The variation of $W^i F_i$ cannot cancel Eq. (3.39), since in

$$\delta_\epsilon \left(W^i F_i\right) \sim \delta_\epsilon W^i F_i + W^i \delta_\epsilon F_i \tag{3.41}$$

the first term will have no derivative, and the second term will have no F-term.

Hence we conclude that $x^{ij} = 0$.

W^{ij} contains only ϕ and no ϕ^* ("holomorphic")

Note that we have proven that $U \equiv x^{ij} \equiv 0$, and consequently $\mathcal{L}_{\text{int}}$ simplifies to

$$\mathcal{L}_{\text{int}} = -\frac{1}{2} W^{ij}(\phi,\phi^*)\psi_i\psi_j + W^i(\phi,\phi^*)F_i. \tag{3.42}$$

Consider its variation:

$$\begin{aligned}\delta_\epsilon \mathcal{L}_{\text{int}} = & -\frac{1}{2}\frac{\delta W^{ij}}{\delta\phi_k}\delta_\epsilon\phi_k(\psi_i\psi_j) - \frac{1}{2}\frac{\delta W^{ij}}{\delta\phi^{*k}}\delta_\epsilon\phi^{*k}(\psi_i\psi_j) - \frac{1}{2}W^{ij}\delta_\epsilon\psi_i\psi_j - \frac{1}{2}W^{ij}\psi_i\delta_\epsilon\psi_j \\ & + \frac{\delta W^i}{\delta\phi_k}\delta_\epsilon\phi_k F_i + \frac{\delta W^i}{\delta\phi^{*k}}\delta_\epsilon\phi^{*k} F_i + W^i\delta_\epsilon F_i \end{aligned} \tag{3.43}$$

Note that the red terms all contain 4 Weyl spinors, the blue terms all contain 1 spacetime derivative, and the green terms are all linear in F. As a consequence, they have to *cancel separately*.

The red terms:

$$
\begin{aligned}
-\frac{1}{2}\frac{\delta W^{ij}}{\delta\phi_k}\delta_\epsilon\phi_k(\psi_i\psi_j) &- \frac{1}{2}\frac{\delta W^{ij}}{\delta\phi^{*k}}\delta_\epsilon\phi^{*k}(\psi_i\psi_j) \\
&= -\frac{1}{2}\frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_k(\psi_i\psi_j) - \frac{1}{2}\frac{\delta W^{ij}}{\delta\phi^{*k}}\epsilon^\dagger\psi^{\dagger k}(\psi_i\psi_j) \\
&= -\frac{1}{2}\frac{1}{3}\left(\frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_k(\psi_i\psi_j) + \frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_k(\psi_i\psi_j) + \frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_k(\psi_i\psi_j)\right) - \frac{1}{2}\frac{\delta W^{ij}}{\delta\phi^{*k}}\epsilon^\dagger\psi^{\dagger k}(\psi_i\psi_j) \\
&= -\frac{1}{2}\frac{1}{3}\left(\frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_k(\psi_i\psi_j) + \frac{\delta W^{ki}}{\delta\phi_j}\epsilon\psi_j(\psi_k\psi_i) + \frac{\delta W^{jk}}{\delta\phi_i}\epsilon\psi_i(\psi_j\psi_k)\right) - \frac{1}{2}\frac{\delta W^{ij}}{\delta\phi^{*k}}\epsilon^\dagger\psi^{\dagger k}(\psi_i\psi_j)
\end{aligned}
\tag{3.44}
$$

It is easy to see that W^{ij} is symmetric in i, j:

$$
W^{ij}\psi_i\psi_j \overset{\text{rename}}{=} W^{ji}\psi_j\psi_i \overset{\text{Eq. (3.16)}}{=} W^{ji}\psi_i\psi_j \quad \rightsquigarrow \quad W^{ji} = W^{ij} \tag{3.45}
$$

If $\delta W^{ij}/\delta\phi_k$ were cyclic in i, j, k, then the term in brackets in Eq. (3.44) would cancel as a consequence of the Fierz identity:

$$
\begin{aligned}
\frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_k(\psi_i\psi_j) &+ \frac{\delta W^{ki}}{\delta\phi_j}\epsilon\psi_j(\psi_k\psi_i) + \frac{\delta W^{jk}}{\delta\phi_i}\epsilon\psi_i(\psi_j\psi_k) \\
&= \frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_k(\psi_i\psi_j) + \frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_j(\psi_k\psi_i) + \frac{\delta W^{ij}}{\delta\phi_k}\epsilon\psi_i(\psi_j\psi_k) \\
&= \frac{\delta W^{ij}}{\delta\phi_k}\epsilon^\alpha\left(\psi_{k\alpha}(\psi_i\psi_j) + \psi_{j\alpha}(\psi_k\psi_i) + \psi_{i\alpha}(\psi_j\psi_k)\right) \qquad \Big|\text{Eq. (3.23)} \\
&= 0
\end{aligned}
\tag{3.46}
$$

Since ϵ and ψ_i, ψ_j, ψ_k are arbitrary, one can see that the converse is also true: If Eq. (3.46) vanishes identically, $\delta W^{ij}/\delta\phi_k$ is cyclic in i, j, k.

Another observation is that there is no Fierz or any other identity that could possible cancel the terms corresponding to $\delta W^{ij}/\delta\phi^{*k}$. These terms have the form $\psi^{\dagger k}\psi_i\psi_j$, and if we permute the indices, every time another field will carry the dagger (note that $i \neq j \neq k$), and a cancellation is impossible unless the fields are real. This is a very important result and justifies to be boxed:

$$
\boxed{W^{ij} \text{ is holomorphic, i.e. } W^{ij} = W^{ij}(\phi) \text{ and } W^{ij} \neq W^{ij}(\phi^*)}
$$

Let us summarize what we know so far. W^{ij} does not depend on ϕ^* and is a polynomial of degree at most 1 (see remarks following Eq. (3.34) on page 19):

$$
W^{ij} = M^{ij} + y^{ijk}\phi_k \tag{3.47}
$$

We can go one step further (we will see later why this is very useful!) and use the fact that W^{ij} is symmetric in i, j and its variation $\delta W^{ij}/\delta\phi_k$ is cyclic in i, j, k:

$$\boxed{W^{ij} = \frac{\partial^2}{\partial\phi_i\partial\phi_j}W} \tag{3.48}$$

with

$$W = L^i\phi_i + \frac{1}{2}M^{ij}\phi_i\phi_j + \frac{1}{6}y^{ijk}\phi_i\phi_j\phi_k. \tag{3.49}$$

W is called the *superpotential*. The reason why we chose to rewrite W^{ij} as the second derivative of W is that we will see that we can also relate W_i to W. Thus, all interactions will be given in terms of a single function W.

W^i is holomorphic and given in terms of the superpotential

Now consider the terms that contain 1 spacetime derivative (blue terms in Eq. (3.43) on page 21):

$$\begin{aligned}
\delta_\epsilon\mathcal{L}_{\text{int}} &\supset -\frac{1}{2}W^{ij}\delta_\epsilon\psi_i\psi_j - \frac{1}{2}W^{ij}\psi_i\delta_\epsilon\psi_j + W^i\delta_\epsilon F_i \\
&= -W^{ij}\delta_\epsilon\psi_i\psi_j + W^i\delta_\epsilon F_i \qquad \Big| W^{ij} \text{ is symmetric} \\
&= -W^{ij}\psi_i^\alpha\delta_\epsilon\psi_{j\alpha} + W^i\delta_\epsilon F_i \qquad \Big| \text{Eq. (3.33)} \\
&= -W^{ij}\psi_i^\alpha(i\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dagger\dot\alpha}\partial_\mu\phi_j + \epsilon_\alpha F_j) + W^i(+i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi) \qquad \Big| \text{Eq. (3.24)} \\
&= -W^{ij}\psi_i^\alpha(i\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dagger\dot\alpha}\partial_\mu\phi_j + \epsilon_\alpha F_j) + W^i(-i\partial_\mu\psi\sigma^\mu\epsilon^\dagger) \qquad \Big| \text{no derivative; drop it} \\
&= -iW^{ij}\partial_\mu\phi_j\psi_i\sigma^\mu\epsilon^\dagger - iW^i\partial_\mu\psi\sigma^\mu\epsilon^\dagger \qquad \Big| \text{Eq. (3.48)} \\
&= -i\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\partial_\mu\phi_j\psi_i\sigma^\mu\epsilon^\dagger - iW^i\partial_\mu\psi\sigma^\mu\epsilon^\dagger \\
&= -i\partial_\mu\left(\frac{\partial W}{\partial\phi_i}\right)\psi_i\sigma^\mu\epsilon^\dagger - iW^i\partial_\mu\psi\sigma^\mu\epsilon^\dagger
\end{aligned} \tag{3.50}$$

This last expression is a total derivative and hence does not contribute to the action, *if and only if*

$$\boxed{W^i = \frac{\partial W}{\partial\phi_i}.} \tag{3.51}$$

This means that we can express W^{ij} and W^i in terms of *the same holomorphic function* W, the superpotential.

The rest of the terms cancel against each other

We now show that all the remaining terms (green) in Eq. (3.43) on page 21 and in Eq. (3.50) on the previous page cancel:

$$
\begin{aligned}
\frac{\delta W^i}{\delta \phi_k}\delta_\epsilon \phi_k F_i + \cancel{\frac{\delta W^j}{\delta \phi^{*k}}}\delta_\epsilon \phi^{*k} F_i + W^{ij}\psi_i \epsilon F_j &= \frac{\delta W^i}{\delta \phi_k}\delta_\epsilon \phi_k F_i + W^{ij}\psi_i \epsilon F_j \quad \Big| W \text{ is holomorphic} \\
&= \frac{\delta W^i}{\delta \phi_k}\epsilon\psi_k F_i + W^{ij}\psi_i \epsilon F_j \quad \Big| \text{Eq. (3.33)} \\
&= \frac{\delta W^i}{\delta \phi_k}\epsilon\psi_k F_i + W^{ij}\epsilon\psi_i F_j \quad \Big| \text{Eq. (3.16)} \\
&= \frac{\partial^2 W}{\partial \phi_i \partial \phi_k}\epsilon\psi_k F_i + W^{ij}\psi_i \epsilon F_j \quad \Big| \text{Eq. (3.51)} \\
&= \frac{\partial^2 W}{\partial \phi_i \partial \phi_k}\epsilon\psi_k F_i + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}\psi_i \epsilon F_j \quad \Big| \text{Eq. (3.48)} \\
&= \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}\epsilon\psi_i F_j + \frac{\partial^2 W}{\partial \phi_i \partial \phi_j}\psi_i \epsilon F_j \quad \Big| k \to j \text{ and } W^{ij} \text{ symmet} \\
&= 0 \qquad (3.52)
\end{aligned}
$$

3.2.5 "Phenomenology" of the Wess-Zumino Model

We will now consider the Lagrangian for a single chiral supermultiplet in the presence of interactions

$$
\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}} \\
&= \partial_\mu \phi^* \partial^\mu \phi - i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi + F^* F - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^*_{ij}\psi^{i\dagger}\psi^{j\dagger}\right) + W^i F_i + W^{i*} F^*_i
\end{aligned} \tag{3.53}
$$

where the most general form for the superpotential W is given by Eq. (3.49) on the preceding page:

$$
W = L^i \phi_i + \frac{1}{2} M^{ij}\phi_i\phi_j + \frac{1}{6} y^{ijk}\phi_i\phi_j\phi_k \tag{3.54}
$$

❶ We can express the F_i's by the W_i's.

Consider the terms in Eq. (3.65) that only contain F or F^*:

$$
\mathcal{L} \;\supset\; F^{i*}F_i + W^i F_i + W^{i*} F^*_i \tag{3.55}
$$

The equations of motion give

$$
\frac{\mathcal{L}}{\partial F_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu F_i)} = 0 \quad \rightsquigarrow \quad \boxed{F^{i*} = -W^i} \tag{3.56}
$$

and

$$\frac{\mathcal{L}}{\partial F^{i*}} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu F^{i*})} = 0 \quad \rightsquigarrow \quad \boxed{F_i = -W_i^*} \tag{3.57}$$

❷ We can now completely eliminate the F_i's and write everything in terms of the superpotential W and its derivatives by substituting Eq. (3.56) and Eq. (3.57) into Eq. (3.65).

$$\begin{aligned}\mathcal{L} &= \partial_\mu\phi^*\partial^\mu\phi - i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + F^*F - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^*_{ij}\psi^{i\dagger}\psi^{j\dagger}\right) + W^iF_i + W^{i*}F_i^* \\ &= \boxed{\partial_\mu\phi^*\partial^\mu\phi - i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^*_{ij}\psi^{i\dagger}\psi^{j\dagger}\right) - W_iW^{i*}}\end{aligned} \tag{3.58}$$

❸ The term W_iW^{i*} in Eq. (3.58) is known as the *scalar potential*:

$$\boxed{V(\phi,\phi^*) = W^iW_i^* = F^{i*}F_i} \tag{3.59}$$

Note that it is always non-negative, since it is a modulus-squared (i.e. $A^*A = |A|^2 \geq 0$ is always true for any $A \in \mathbb{C}$). As such, it is always bounded from below.

We will write out explicitly the scalar potential for later purposes (disregard L_i since it is not present in SM):

$$\begin{aligned}V = W^iW_i^* &= \left(\cancel{L^i} + M^{ij}\phi_j + \frac{1}{2}y^{ijk}\phi_j\phi_k\right)\left(\cancel{L_i^*} + M^*_{im}\phi^{m*} + \frac{1}{2}y^*_{imn}\phi^{m*}\phi^{n*}\right) \\ &= M^{ij}\phi_jM^*_{im}\phi^{m*} + \frac{1}{2}y^{ijk}\phi_j\phi_kM^*_{im}\phi^{m*} + M^{ij}\phi_j\frac{1}{2}y^*_{imn}\phi^{m*}\phi^{n*} + \frac{1}{4}y^{ijk}\phi_j\phi_ky^*_{imn}\phi^{m*}\phi^{n*} \\ &= M^{ij}M^*_{im}\phi_j\phi^{m*} + \frac{1}{2}y^{ijk}M^*_{im}\phi_j\phi_k\phi^{m*} + \frac{1}{2}M^{ij}y^*_{imn}\phi_j\phi^{m*}\phi^{n*} + \frac{1}{4}y^{ijk}y^*_{imn}\phi_j\phi_k\phi^{m*}\phi^{n*}\end{aligned} \tag{3.60}$$

This part of the Lagrangian gives the interaction of the scalars among each other and determines their masses. Later, in the MSSM, the Higgs potential will be part of the scalar potential. Note that it is completely determined by the F-term. That's the reason why in the MSSM the Higgs mass can be *calculated* and is not a free parameter like in the SM.

❹ What about the interaction of the fermions?

The relevant part of the Lagrangian is

$$\begin{aligned}\mathcal{L} &\supset -\frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^*_{ij}\psi^{i\dagger}\psi^{j\dagger}\right) \\ &= -\frac{1}{2}\left((M^{ij} + y^{ijk}\phi_k)\psi_i\psi_j + (M^*_{ij} + y^*_{ijk}\phi^{k*})\psi^{i\dagger}\psi^{j\dagger}\right) \\ &= -\frac{1}{2}M^{ij}\psi_i\psi_j - \frac{1}{2}y^{ijk}\phi_k\psi_i\psi_j - \frac{1}{2}M^*_{ij}\psi^{i\dagger}\psi^{j\dagger} - \frac{1}{2}y^*_{ijk}\phi^{k*}\psi^{i\dagger}\psi^{j\dagger}\end{aligned} \tag{3.61}$$

So we obtain mass terms for the fermions and the cubic fermion-fermion-scalar Yukawa interactions.

❺ Now we want to show that the scalar and fermion masses are equal. Consider the equation of motion for the scalar fields ϕ_i:

$$\frac{\partial\mathcal{L}}{\partial\phi^{m*}}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{m*})}=0 \quad\rightsquigarrow\quad M^{ij}M^*_{im}\phi_j-\partial_\mu\partial^\mu\phi_m=0 \quad\rightsquigarrow\quad \partial_\mu\partial^\mu\phi_m=M^{ij}M^*_{im}\phi_j+\dots \tag{3.62}$$

In other words,

$$\partial_\mu\partial^\mu\phi_m=M^{ij}M^*_{im}\phi_j \quad\leftrightarrow\quad \partial_\mu\partial^\mu\phi_m=M^*_{mi}M^{ij}\phi_j \quad\leftrightarrow\quad \partial_\mu\partial^\mu\phi_m=(M^*M)_m{}^j\phi_j \tag{3.63}$$

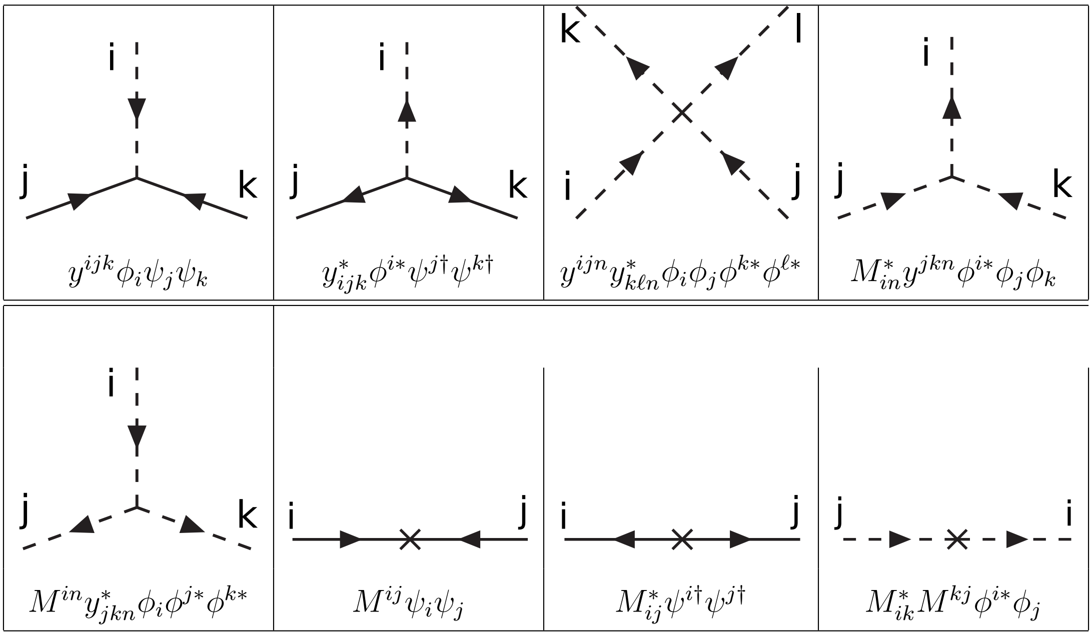
This corresponds to scalar fields with mass matrix M^*M (remember the Klein-Gordan equation $(\Box+m^2)\phi=0$). Now consider the equation of motion for the Weyl fermions.

$$\frac{\partial\mathcal{L}}{\partial\psi^{k\dagger}}-\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi^{k\dagger})}=0 \quad\rightsquigarrow\quad i\overline{\sigma}^\mu\partial_\mu\psi_k-W^*_{ik}\psi^{i\dagger}=0 \quad\rightsquigarrow\quad i\overline{\sigma}^\mu\partial_\mu\psi_k=M^*_{ki}\psi^{i\dagger} \tag{3.64}$$

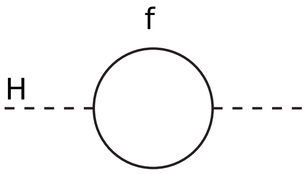
Remember that one cannot write down a mass term for a *single* Weyl spinor; we always need a left-handed and right-handed one, and that's why we have the *hermitian conjugate* spinor on the right-hand side of the equation. Comparing Eq. (3.64) with the Dirac equation in the chiral basis ($i\overline{\sigma}^\mu\partial_\mu\psi_L=m\psi_R$ and $i\sigma^\mu\partial_\mu\psi_R=m\psi_L$), we conclude that the mass matrix (for the Dirac spinor corresponding to 2 Weyl spinors) is indeed M_{ki}. As a consequence, the scalar and fermion mass matrices are equal.

❻ So what do the interactions look like? For ease of reference, we reproduce in the following lines the full interacting Lagrangian of the simplest supersymmetric model, i.e. we takeEq. (3.58) and substitute the expression for W^{ij} in Eq. (3.61) and for the scalar potential in Eq. (3.82):

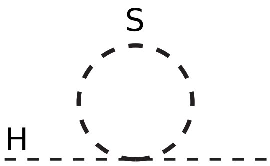
$$\begin{aligned}\mathcal{L}&=\mathcal{L}_{\text{free}}+\mathcal{L}_{\text{int}}\\&=\partial_\mu\phi^*\partial^\mu\phi-i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi-\frac{1}{2}\left(W^{ij}\psi_i\psi_j+W^*_{ij}\psi^{i\dagger}\psi^{j\dagger}\right)-W_iW^{i*}\\&=\partial_\mu\phi^*\partial^\mu\phi-i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi-\frac{1}{2}M^{ij}\psi_i\psi_j-\frac{1}{2}y^{ijk}\phi_k\psi_i\psi_j-\frac{1}{2}M^*_{ij}\psi^{i\dagger}\psi^{j\dagger}-\frac{1}{2}y^*_{ijk}\phi^{k*}\psi^{i\dagger}\psi^{j\dagger}\\&\quad+M^{ij}M^*_{im}\phi_j\phi^{m*}+\frac{1}{2}y^{ijk}M^*_{im}\phi_j\phi_k\phi^{m*}+\frac{1}{2}M^{ij}y^*_{imn}\phi_j\phi^{m*}\phi^{n*}+\frac{1}{4}y^{ijk}y^*_{imn}\phi_j\phi_k\phi^{m*}\phi^{n*}\end{aligned} \tag{3.65}$$



❼ Now remember the problem with the quadratic divergences.



$$\Delta m_H^2 \;=\; -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{\rm UV}^2 + \ldots$$



$$\Delta m_H^2 \;=\; \frac{\lambda_S}{16\pi^2}\left[\Lambda_{\rm UV}^2 - 2m_S^2\log(\Lambda_{\rm UV}/m_S) + \ldots\right]$$

The magic of SUSY is that $\lambda_f = y^{ijk}$ and $\lambda_S = y^{ijn}y^*_{k\ell n}$ (see table above), and this is exactly what is needed for the quadratic divergence to cancel!

3.2.6 Summary

Let us pause for a moment and summarize what we have learned so far.

- The simplest supersymmetric model (Wess-Zumino model) without interactions is given by

$$\boxed{\begin{aligned}&\mathcal{L}_{\text{free}} = \partial_\mu\phi^*\partial^\mu\phi - i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi\\ &\delta\phi = \epsilon\psi, \quad \delta\psi_\alpha = +i\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dagger\dot\alpha}\partial_\mu\phi\end{aligned}} \tag{3.66}$$

- If you insist that the SUSY algebra should close off-shell, you need to introduce an auxiliary field:

$$\boxed{\begin{aligned}&\mathcal{L}_{\text{free}} = \partial_\mu\phi^*\partial^\mu\phi - i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi + F^*F\\ &\delta\phi = \epsilon\psi, \quad \delta\psi_\alpha = +i\sigma^\mu_{\alpha\dot\alpha}\epsilon^{\dagger\dot\alpha}\partial_\mu\phi + \epsilon_\alpha F, \quad \delta F = +i\epsilon^\dagger\overline{\sigma}^\mu\partial_\mu\psi\end{aligned}} \tag{3.67}$$

- Note that for <u>each</u> Weyl spinor, you have to introduce <u>one complex</u> scalar.

- <u>All interactions</u> can be described in terms of a single, <u>holomorphic</u> function that we call the superpotential. This includes <u>masses</u> for the fermions and bosons.

- The full Lagrangian with interactions takes the form:

$$\boxed{\begin{aligned}&\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - i\psi^\dagger\overline{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^*_{ij}\psi^{i\dagger}\psi^{j\dagger}\right) - W^iW^*_i\\ &W^i = \frac{\partial W}{\partial\phi_i}, \quad W^{ij} = \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\end{aligned}} \tag{3.68}$$

- The part of the Lagrangian which contains only scalar fields is called the *scalar potential*:

$$\boxed{V(\phi,\phi^*) = W^iW^*_i = F^{i*}F_i} \tag{3.69}$$

- The <u>masses</u> of the fermions and bosons are <u>exactly equal</u>. This is in stark contradiction to experiment, so if SUSY is realized in Nature, it must be broken.

- The scalar quartic coupling is not only given/determined/fixed by the Yukawa couplings, but also fixed at such a value that the quadratic divergence $m_H \sim \Lambda^2$ disappears. This is the strongest motivation for SUSY.

- Warning: The superpotential W is not a potential. It is also not a Lagrangian or part of a Lagrangian. The superpotential is simply an auxiliary construction that allows us to write down the terms which do contribute to the Lagrangian/potential where the prescription is given in Eq. (3.68).

3.3 The Vector Supermultiplet

We do not have the time to cover the vector supermultiplet in any detail, so I will only give the results and point out the analogy to the case of a chiral supermultiplet which we have considered in Section 3.2 on page 13. I hope this will give you "a good feeling" and some "intuition" when dealing with vector supermultiplets. Unfortunately, this is all I can do in only 8 hours dedicated to SUSY.

SUSY associates a boson to a fermion and vice versa, so it is not too surprising that for a gauge field A_μ, we introduce a Weyl spinor λ_α (called "gaugino"). If the gauge group $\mathfrak{G}$ is more complicated than U(1), then we need another index to enumerate the gauge bosons and Weyl spinors, i.e. we have A_μ^a and λ_α^a, where $a = 1, \ldots, \dim(\mathfrak{G})$. E.g. for SU(2), $a = 1, 2, 3$ for the three Pauli matrices. It is clear what Lagrangian we should write down:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - i\lambda^{a\dagger}\overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2} D^a D^a \tag{3.70}$$

This is in complete analogy to Eq. (3.67) on the preceding page where we of course had to substitute the kinetic term for a scalar by the kinetic term of a vector field; the expression for a Weyl fermion stays the same, but we substituted $\partial_\mu \to D_\mu$, since we want the Lagrangian to be gauge invarant (incidentally, this couples A_μ^a to λ_α^a); and we added an auxiliary field D^a so that the SUSY algebra closes off-shell just like we had introduced the F-term before. Not surprisingly, it is called a D-term.

Now, if you count the *off-shell* bosonic and fermionic degrees of freedom (d.o.f.), you find the following: λ^a (for a fixed) corresponds to 2 complex = 4 real d.o.f. whereas A_μ^a has 3 real d.o.f., since 1 d.o.f. is lost due to gauge fixing (e.g. Coulomb gauge $\partial_\mu A^{a\mu} = 0$). We know that in SUSY we have equal bosonic and fermionic degrees of freedom, so we conclude that D^a should correspond to 1 d.o.f. and hence be *real*:

$$\boxed{(D^a)^* = D^a} \tag{3.71}$$

We can now do some guesswork like before and find that the Lagrangian in Eq. (3.70) is invariant under the following SUSY transformations:

$$\delta A_\mu^a = -\frac{1}{\sqrt{2}} \left(\epsilon^\dagger \overline{\sigma}_\mu \lambda^a + \lambda^{\dagger a} \overline{\sigma}_\mu \epsilon \right) \tag{3.72}$$

$$\delta \lambda_\alpha^a = \frac{i}{2\sqrt{2}} (\sigma^\mu \overline{\sigma}^\nu \epsilon)_\alpha \, F_{\mu\nu}^a + \frac{1}{\sqrt{2}} \epsilon_\alpha \, D^a \tag{3.73}$$

$$\delta D^a = \frac{i}{\sqrt{2}} \left(\epsilon^\dagger \overline{\sigma}^\mu D_\mu \lambda^a - D_\mu \lambda^{\dagger a} \overline{\sigma}^\mu \epsilon \right) \tag{3.74}$$

Of course, we will again have to check that the SUSY algebra closes (in analogy to Section 3.2.2 on page 16), and it turns out that it does.

So far so good. Now we want to add the matter fields, i.e. the ψ_k and ϕ_k which will later correspond to the electrons and selectrons. It turns out that we have to do that in three steps:

❶ Do the obvious, i.e. introduce the Lagrangian from Section 3.2 for "matter fields":

$$\mathcal{L}_{\text{chiral}} = \partial_\mu \phi^* \partial^\mu \phi - i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^*_{ij}\psi^{i\dagger}\psi^{j\dagger}\right) - W^i W^*_i \tag{3.75}$$

❷ This is also straightforward, namely couple matter to vector bosons by $\partial_\mu \to D_\mu$!

❸ It turns out that this does not suffice, and we are missing interactions of the form ψ-ϕ-λ. We are forced to introduce the following interactions by hand:

$$(\phi^* T^a \psi)\lambda^a, \qquad \lambda^{\dagger a}(\psi^\dagger T^a \phi), \qquad (\phi^* T^a \phi) D^a \tag{3.76}$$

Compared to what happened in the Standard Model, this is not exactly elegant. But stay tuned, it gets worse. Since we have introduced these extra terms to ensure invariance under SUSY, we also need to modify the SUSY transformations of the chiral supermultiplet:

$$\delta\phi_i = \epsilon\psi_i \tag{3.77}$$

$$\delta\psi_{i\alpha} = +i(\sigma^\mu \epsilon^\dagger)_\alpha\, D_\mu \phi_i + \epsilon_\alpha F_i \tag{3.78}$$

$$\delta F_i = +i\epsilon^\dagger \bar{\sigma}^\mu D_\mu \psi_i \;+\; \sqrt{2}g(T^a\phi)_i\, \epsilon^\dagger \lambda^{\dagger a} \tag{3.79}$$

The full Lagrangian is now:

$$\boxed{\begin{aligned} \mathcal{L}_{\text{full}} = D_\mu \phi^* D^\mu \phi - i\psi^\dagger \bar{\sigma}^\mu D_\mu \psi - \frac{1}{2}\left(W^{ij}\psi_i\psi_j + W^*_{ij}\psi^{i\dagger}\psi^{j\dagger}\right) - W^i W^*_i &\quad \leftarrow \quad \mathcal{L}_{\text{chiral}} \\ - \frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - i\lambda^{a\dagger}\bar{\sigma} D_\mu \lambda^a + \frac{1}{2}D^a D^a &\quad \leftarrow \quad \mathcal{L}_{\text{gauge}} \\ - \sqrt{2}g(\phi^* T^a \psi)\lambda^a - \sqrt{2}g\lambda^{\dagger a}(\psi^\dagger T^a \phi) + g(\phi^* T^a \phi)D^a &\quad \leftarrow \quad \mathcal{L}_{\text{extra}} \end{aligned}} \tag{3.80}$$

As before, we can eliminate D^a:

$$\frac{\partial \mathcal{L}}{\partial D^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial D^a} = 0 \quad \rightsquigarrow \quad D^a = -g(\phi^* T^a \phi) \tag{3.81}$$

Just like the F-term, the D-term contains only scalars, and consequently it contributes to the scalar potential:

$$\boxed{V(\phi, \phi^*) = F^{i*}F_i + \sum_a D^a D^a = W^*_i W^i + \sum_a g_a^2 (\phi^* T^a \phi)^2} \tag{3.82}$$

These two contributions are called the F-term and D-term contributions, respectively (just that you learn the nomenclature). Note that there are as many D-terms as there are generators in the algebra (i.e. the dimension of the algebra).

Now you can see/guess/intuitively understand why in SUSY the Higgs mass is a *prediction*. The Higgs will be part of the scalar potential, and the scalar potential is fully determined by the F- and D-terms, i.e. by the Yukawa couplings, masses, and gauge couplings.

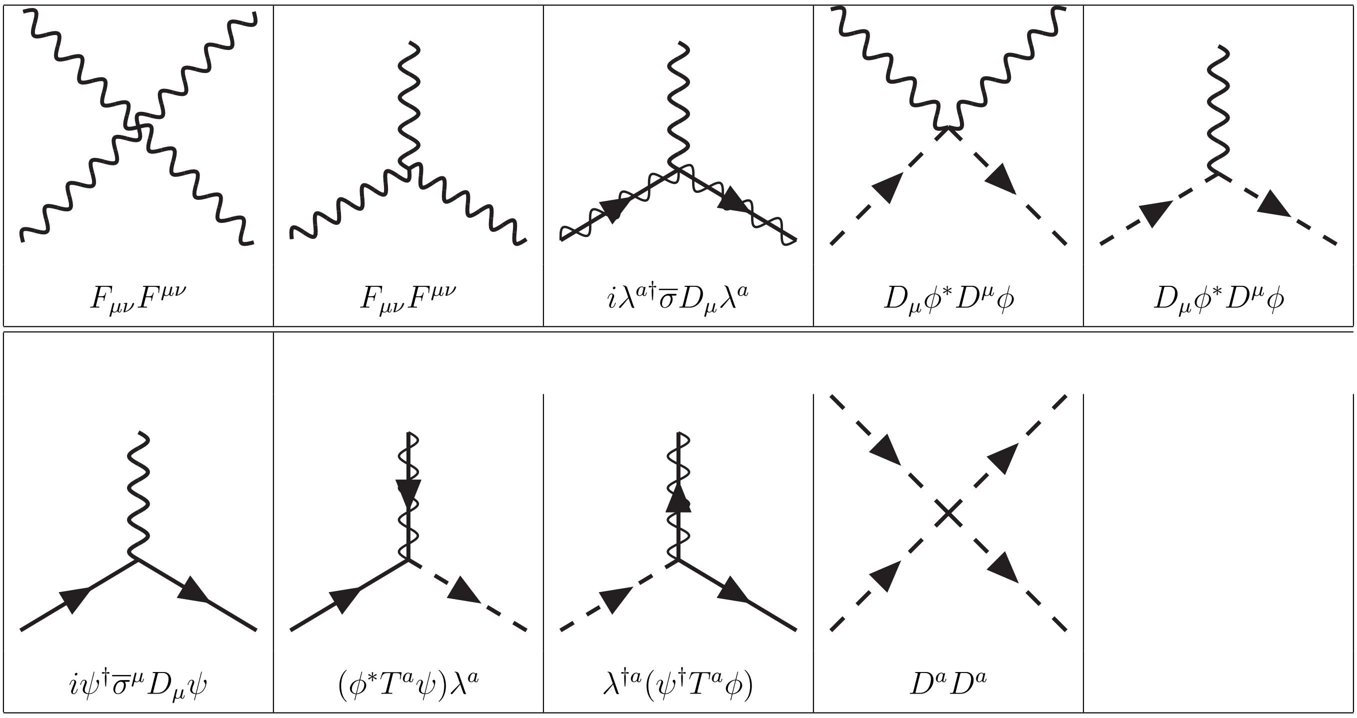
We will now consider the interactions that follow from the Lagrangian in Eq. (3.80). For convenience, we reproduce some previously derived expressions so that the Feynman graphs are easier to understand:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^a_\mu A^b_\nu$$

$$D_\mu \phi = \partial_\mu \phi - ig A^a_\mu T^a_R \phi, \qquad \text{e.g. } T^a_R = \frac{\sigma^a}{2}, \;\; a = 1, 2, 3$$

$$D_\mu \psi = \partial_\mu \psi - ig A^a_\mu T^a_R \psi, \qquad \text{e.g. } T^a_R = \frac{\lambda^a}{2}, \;\; a = 1, \ldots, 8$$

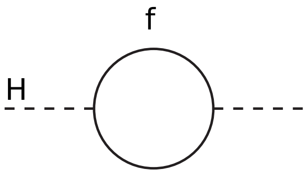
$$D_\mu \lambda^a = \partial_\mu \lambda^a - ig f^{abc} A^b_\mu \lambda^c, \qquad \text{always (!) } T^a_R = (f^a)_{bc} = f^{abc}$$



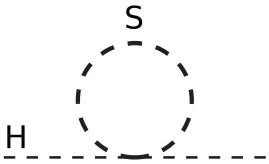
4 Which Problems does SUSY Solve?

4.1 SUSY: Pros

❶ Hierarchy problem is "solved"



$$\Delta m_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{\rm UV}^2 + \ldots$$



$$\Delta m_H^2 = \frac{\lambda_S}{16\pi^2}\left[\Lambda_{\rm UV}^2 - 2m_S^2\log(\Lambda_{\rm UV}/m_S) + \ldots\right]$$

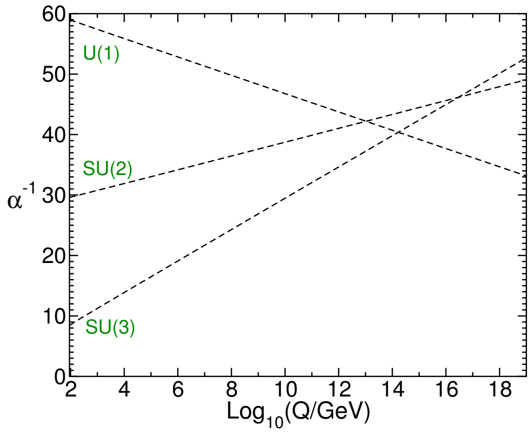
➢ $\lambda_S = |\lambda_f|^2$ and introduces 2 complex scalars for each Dirac fermion

➢ Quadratic divergences cancel

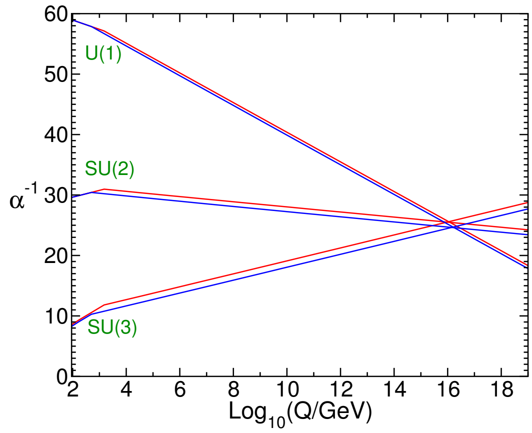
❷ Gauge coupling unification

➢ This is an argument in favor of SUSY <u>and</u> GUTS!

➢ Requires superparticle masses around 1 TeV? No, not sensitive to scalar masses!



NO SUSY



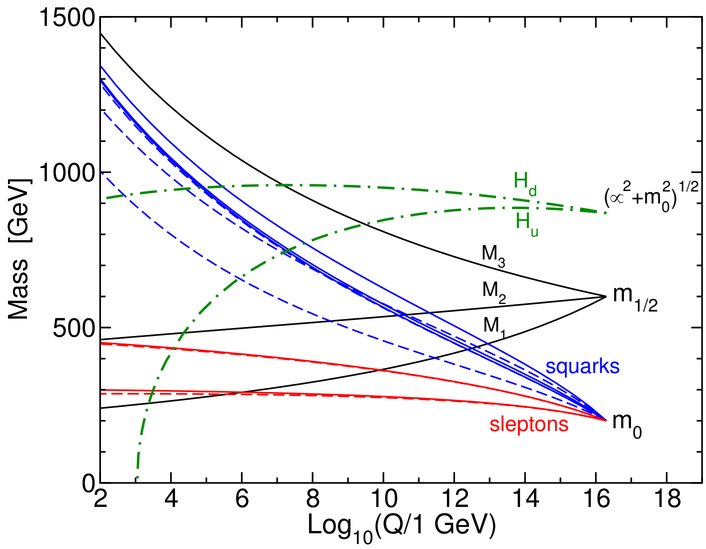
SUSY

4.2 SUSY: Cons

- Predicts equal superpartner masses ⇝ Needs SUSY breaking to be viable
- Predicts fast proton decay ⇝ Needs R-parity to be viable
- SUSY breaking not understood
  - Allow only soft terms, i.e. do not re-introduce hierarchy problem. Circular reasoning?

$$\Delta m_H^2 \; = \; m_{\rm soft}^2 \left[\frac{\lambda}{16\pi^2} \log(\Lambda_{\rm UV}/m_{\rm soft}) + \ldots \right]$$

  - Predictivity to some extent lost: SM → MSSM introduces 105 new parameters



-
- Experiment ⇝ might be fine-tuned after all

4.3 Revisiting the "Problems of Modern Physics"

❶ Too many free parameters

Gauge sector: 3 couplings g', g, g_3	3
Quark sector: 6 masses, 3 mixing angles, 1 CP phase	10
Lepton sector: 6 masses, 3 mixing angles and 1+2 phases	10+2
Higgs sector: Quartic coupling λ and vev v	2
$\boldsymbol{\theta}$ parameter of QCD	1
	26+2

❷ Structure of gauge symmetry

Why the product structure $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$?

Why 3 different coupling constants g', g, g_3?

❸ Structure of family multiplets

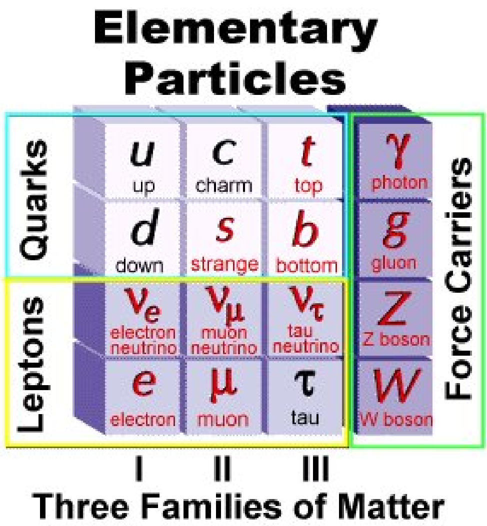
One family is

$$\underset{Q}{(\mathbf{3},\mathbf{2})_{1/3}} + \underset{\bar{u}}{(\overline{\mathbf{3}},\mathbf{1})_{-4/3}} + \underset{\bar{e}}{(\mathbf{1},\mathbf{1})_{-2}} + \underset{\bar{d}}{(\overline{\mathbf{3}},\mathbf{1})_{2/3}} + \underset{L}{(\mathbf{1},\mathbf{2})_{-1}} + \underset{\bar{\nu}}{(\mathbf{1},\mathbf{1})_{0}}$$

Can the particles be reorganized in a single representation?

❹ Repetition of families

Why is this pattern for 1 generation replicated 3 times?



❺ Mass hierarchies and texture of Yukawa couplings

up-quark mass $\sim 2 \times 10^{-3}$ GeV $\quad\leftrightarrow\quad$ top-quark mass ~ 172.3 GeV

Yukawa coupling of top ~ 1, but why are the other quarks so light?

Minimal mixing in quark sector

$$V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \simeq \begin{pmatrix} 0.97 & 0.22 & 0.00 \\ 0.22 & 0.97 & 0.04 \\ 0.00 & 0.04 & 0.99 \end{pmatrix}$$

❻ Light neutrinos and texture of Yukawa couplings

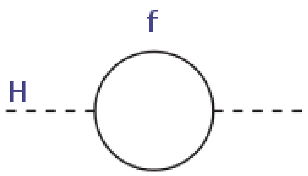
Why are neutrinos so light?

$$\Delta m_\nu^2 \sim 10^{-2} - 10^{-5}\ \text{eV}, \quad \sum m_\nu \lesssim 2\ \text{eV}$$

Maximal mixing in lepton sector

$$U_{\text{PMNS}} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \simeq \begin{pmatrix} 0.8 & 0.5 & 0.0 \\ -0.4 & 0.6 & 0.7 \\ 0.4 & -0.6 & 0.7 \end{pmatrix}$$

❼ Hierarchy problem



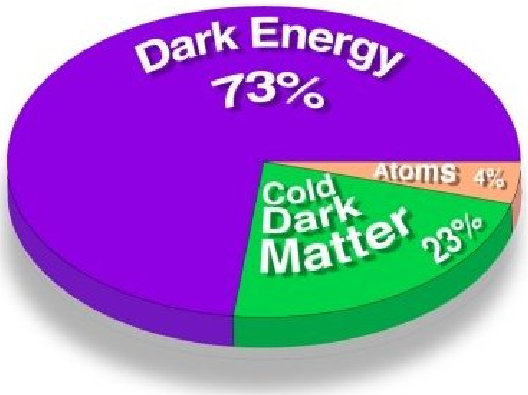
$$\Delta m_H^2 = -\frac{|\lambda_f|^2}{8\pi^2}\Lambda_{\text{UV}}^2 + \ldots$$

- Higgs mass is quadratically divergent
- Standard Model is renormalizable and infinities can be absorbed into a finite number of physical parameters
- Hierarchy problem arises if one goes beyond renormalizability
  $\rightsquigarrow$ Cut-off Λ_{UV} acquires physical meaning

- Higgs mass is dragged to cut-off scale e.g. $\Lambda_{UV} \sim M_{\text{Planck}}$
- However, we need a light Higgs $\mathcal{O}(100)$ GeV
- Analogous problems arise from presence of any heavy particle

❽ Dark Matter and Dark Energy

23% of our universe is made up of dark matter and the Standard Model offers no candidate particle . . .



73% of our universe is made up of dark energy and the cosmological constant as calculated from QFT is the worst-predicted quantity in particle physics

❾ Gravity

- Scales relevant in everyday life ⇝ Newton's theory
- Satellites, solar system, etc. ⇝ Still Newton's theory
- Cosmological scales ⇝ Einstein's theory of GR
- Very small scales ⇝ Need quantum theory of gravitation
- Don't know how to quantize gravity and how to unify with SM

  → String theory or loop quantum gravity

❿ Many other problems

Baryon asymmetry in the universe, charge quantization, . . .

Acknowledgments

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Bibliography

[1] H. Kalka and G. Soff. Supersymmetry. (In German). 1997.

[2] M. Maggiore. A Modern introduction to quantum field theory. 2005.

[3] S. P. Martin. A Supersymmetry primer. 1997.

[4] H. Muller-Kirsten and A. Wiedemann. Supersymmetry: An Introduction with Conceptual and Calculational Details. 1986.