M2R PSA **10 février 2011, durée 3h00**
Physics beyond the standard model
All documents allowed

PART I: Supersymmetry

Formulae:

Space-time metric:

- $\eta^{\mu\nu} = \text{diag}[1,-1,-1,-1]$

In the Weyl representation:

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{\alpha\dot\alpha} \\ (\bar\sigma^\mu)^{\dot\alpha\alpha} & 0 \end{pmatrix}, \; \gamma^5 = \begin{pmatrix} \delta^\alpha_\beta & 0 \\ 0 & \delta^{\dot\alpha}_{\dot\beta} \end{pmatrix}, \; \frac{1}{2}\Sigma^{\mu\nu} = \begin{pmatrix} i(\sigma^{\mu\nu})_\alpha^{\ \beta} & 0 \\ 0 & i(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \end{pmatrix}$$

with

$$(\sigma^\mu)_{\alpha\dot\alpha} = (I_2, \vec\sigma)\,, \; (\bar\sigma^\mu)^{\dot\alpha\alpha} = (I_2, -\vec\sigma)$$

where $\vec\sigma = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauil matrices. Furthermore,

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{1}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)\,, \; (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{1}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)\,.$$

Some spinor relations:

- $\chi\psi := \chi^\alpha\psi_\alpha = \psi\chi$, $\bar\chi\bar\psi := \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\psi\bar\chi$, $(\chi\psi)^\dagger = \psi^\dagger\chi^\dagger = \bar\psi\bar\chi$
- $\chi\sigma^\mu\bar\psi := \chi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\psi^{\dot\alpha}$, $\bar\chi\bar\sigma^\mu\psi := \bar\chi_{\dot\alpha}\bar\sigma^{\mu\,\dot\alpha\alpha}\psi_\alpha$
- $(\chi\sigma^\mu\bar\psi)^\dagger = \psi\sigma^\mu\bar\chi$
- $\chi\sigma^\mu\bar\psi = -\bar\psi\bar\sigma^\mu\chi$
- $\chi\sigma^{\mu\nu}\psi = -\psi\sigma^{\mu\nu}\chi$, $\bar\chi\bar\sigma^{\mu\nu}\bar\psi = -\bar\psi\bar\sigma^{\mu\nu}\bar\chi$
- $(\chi\sigma^{\mu\nu}\psi)^\dagger = \bar\chi\bar\sigma^{\mu\nu}\bar\psi$

Problem 1: Variations on the SUSY algebra

a) Write down the ($N = 1$) SUSY algebra in terms of the 10 bosonic generators of the Poincaré group (P^μ, $J^{\mu\nu}$) and the 4 fermionic generators (Q_α, and $\bar{Q}_{\dot{\alpha}}$). Here Q_α ($\alpha = 1, 2$), and $\bar{Q}_{\dot{\alpha}}$ ($\dot{\alpha} = \dot{1}, \dot{2}$) are 2-component Weyl spinors transforming according to the $(1/2, 0)$ and $(0, 1/2)$-representations of the Lorentz group, respectively.

b) In 4-component notation we define the Majorana spinor $Q_{M,a}$ ($a = 1, 2, \dot{1}, \dot{2}$):

$$Q_M = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot{\alpha}} \end{pmatrix} .$$

Show that the Dirac-adjoint spinor is given by $\bar{Q}_M = (Q^\beta, \bar{Q}_{\dot{\beta}})$.

c) Show that

$$\begin{aligned} [P^\mu, Q_M] &= 0\,, \\ [J^{\mu\nu}, Q_M] &= -\frac{1}{2}\,\Sigma^{\mu\nu}\,Q_M\,, \\ \{Q_M, \bar{Q}_M\} &= 2\gamma^\mu P_\mu \end{aligned}$$

with γ^μ and $\Sigma^{\mu\nu} := \frac{i}{4}[\gamma^\mu, \gamma^\nu]$ in the Weyl representation.
Hint: Consider $\{Q_{M,a}, \bar{Q}_{M,b}\}$ for fixed $a, b = 1, 2, \dot{1}, \dot{2}$.

d) The SUSY algebra can be expressed entirely in terms of commutators with the help of anti-commuting parameters ξ^α, $\bar{\xi}_{\dot{\alpha}}$ and η^α, $\bar{\eta}_{\dot{\alpha}}$:

$$\{\xi^\alpha, \xi^\beta\} = \{\xi^\alpha, Q_\beta\} = \ldots = [P_\mu, \xi^\alpha] = 0\,.$$

Evaluate the following commutators using the algebra in a):

(i) $[P_\mu, \xi Q] = [P_\mu, \bar{\xi}\bar{Q}]$

(ii) $[J^{\mu\nu}, \xi Q]$, $[J^{\mu\nu}, \bar{\xi}\bar{Q}]$

(iii) $[\xi Q, \eta Q] = [\bar{\xi}\bar{Q}, \bar{\eta}\bar{Q}]$

(iv) $[\xi Q, \bar{\eta}\bar{Q}]$

where we use the summation convention $\xi Q := \xi^\alpha Q_\alpha$, $\bar{\xi}\bar{Q} := \bar{\xi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$.

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Problem 2: A supersymmetric Lagrangian

a) Write down the Wess-Zumino Lagrangian for massless ($m = 0$) and non-interacting ($g = 0$) fields A and ψ

b) In this case ($m = 0, g = 0$) the SUSY transformations are given by

$$\begin{aligned} \delta_\xi A &= \sqrt{2}\, \xi\psi\,, \\ \delta_\xi \psi &= -i\sqrt{2}\, \sigma^\mu \bar{\xi}\, \partial_\mu A \end{aligned}$$

Here, we use again the summation convention and free indices α have been suppressed. Write down the transformation rules for the conjugate fields which can be obtained by hermitian conjugation:

(i) $\delta_\xi A^* = (\delta_\xi A)^\dagger = \ldots$

(ii) $\delta_\xi \bar{\psi} = (\delta_\xi \psi)^\dagger = \ldots$

c) Show that

$$\frac{1}{\sqrt{2}}\delta_\xi \left[(\partial_\mu A^*)(\partial^\mu A)\right] = -\bar{\xi}\bar{\psi}\, \Box A - \xi\psi\, \Box A^* + \partial_\mu K_1^\mu$$

with $\Box := \partial_\mu \partial^\mu$. Specify K_1^μ.

d) Show that

$$\frac{1}{\sqrt{2}}\delta_\xi \left[i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi\right] = \bar{\xi}\bar{\psi}\, \Box A + \xi\psi\, \Box A^* + \partial_\mu K_2^\mu$$

Specify K_2^μ.
Hint: $\sigma^\mu \bar{\sigma}^\nu = \eta^{\mu\nu} + 2\sigma^{\mu\nu}$, $\bar{\sigma}^\mu \sigma^\nu = \eta^{\mu\nu} + 2\bar{\sigma}^{\mu\nu}$ and $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$, $\bar{\sigma}^{\mu\nu} = -\bar{\sigma}^{\nu\mu}$.

e) Conclude that the action is invariant under this symmetry: $\delta_\xi S = 0$.