QCD and collider physics - Problem sheet

Problem 1

Let T_a be Hermitian ($T_a^\dagger = T_a$) and traceless ($\mathrm{T}r(T_a) = 0$) $n \times n$ matrices. Show that there are $n^2 - 1$ linearly independent matrices and that the structure constants f_{abc} in $[T_a, T_b] = i f_{abc} T_c$ are real.

Problem 2

Show that $[T_a, T_b] = i f_{abc} T_c$ and $\mathrm{T}r(T_a T_b) = \frac{1}{2}\delta_{ab}$ lead to $f_{abc} = -f_{acb}$.
Hint : Calculate $\mathrm{T}r(T_c[T_a, T_b])$ and take into account that $\mathrm{T}r(ABC) = \mathrm{T}r(CAB)$.

Problem 3

a) Prove the Fierz identity

$$\sum_{a=1}^{n^2-1} (T_a)_{ij}(T_a)_{kl} = \frac{1}{2}(\delta_{il}\delta_{jk} - \frac{1}{n}\delta_{ij}\delta_{kl})$$

Hint : Use $\mathrm{T}r(T_a T_b) = \frac{1}{2}\delta_{ab}$ in order to calculate m_0 and m_a in

$$M = m_0 \mathbf{1}_n + \sum_{a=1}^{n^2-1} m_a T_a$$

where M is an arbitrary Hermitian $n \times n$ matrix.

b) Prove

$$\sum_{a=1}^{n^2-1} T_a T_a = C_F \mathbf{1}_n \quad \text{with} \quad C_F \equiv \frac{n^2-1}{2n}$$

Problem 4

The generators (X_a) of the adjoint representation of the Lie algebra su(n) can be defined as $(X_a)_{bc} := -i f_{abc}$. Show that the generators respect the algebra :

$$[X_a, X_b] = i f_{abc} X_c$$

Hint : Use $[T_a, T_b] = i f_{abc} T_c$ and the Jacobi identity

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

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Problem 5

Calculate λ in

$$\sum_{c,d=1}^{n^2-1} f_{acd} f_{bcd} = \lambda \delta_{ab}$$

using

$$\sum_{a,b=1}^{n^2-1} X_a X_a = C_A \mathbf{1}_{n^2-1}$$

with $C_A \equiv n$. Show in addition that

$$\begin{aligned}
\sum_{a=1}^{n^2-1} T_a T_b T_a &= \left(C_F - \frac{C_A}{2} \right) T_b \\
\sum_{a,b=1}^{n^2-1} f_{abc} T_a T_b &= i \frac{C_A}{2} T_c
\end{aligned}$$

Problem 6

Confirm the results of problems 3, 4, and 5 for the group $\mathrm{SU}(2)$ by direct calculation. In this case $f_{abc} = \epsilon_{abc}$ and $T_a = \frac{1}{2}\sigma_a$ with the Pauli matrices :

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$