1)

a)

$$d\tilde{G}_{ij}(P_i,P_j) = \sum_{k,l} \int_0^1 dx_1 \int_0^1 dx_2 \; \Gamma_{i\to k}(x_1,\mu_F^2,\mu^2)\,\Gamma_{j\to l}(x_2,\mu_F^2,\mu^2)\; d\hat{G}_{kl}(x_1P_i,x_2P_j,\mu_F^2)$$

↑ somme sur tous les processus possibles

P_i, P_j : les 4-impulsions des partons i, j

x_1P_i, x_2P_j : les 4-impulsions des partons k, l

b)

$$\Gamma_{i\to k}(z,\mu_F^2,\mu^2) = \Gamma^{(0)}_{i\to k} + \Gamma^{(1)}_{i\to k} + \Gamma^{(2)}_{i\to k} + \dots$$

avec $\Gamma^{(0)}_{i\to k} = \delta_{ik}\,\delta(1-z)$

$$\Gamma^{(1)}_{i\to k} = \frac{\alpha_s}{2\pi}\left[P_{i\to k}\,\frac{2}{\epsilon} + F_{i\to k}\right]$$

$$F^{\overline{MS}}_{i\to k}(z,\mu_F^2,\mu^2) = P^{(0)}_{i\to k}(z)\left(\gamma_E - \ln 4\pi + \ln\frac{\mu_F^2}{\mu^2}\right)$$

c)

$$d\tilde{G}_{ij} = \Gamma_{i\to l}\otimes\Gamma_{j\to m}\otimes d\hat{G}_{lm} = d\tilde{G}^{(0)}_{ij} + d\tilde{G}^{(1)}_{ij} + d\tilde{G}^{(2)}_{ij} + \dots$$

$$= \left(\delta_{il}\,\delta(1-x_1) + \Gamma^{(1)}_{i\to l} + \Gamma^{(2)}_{i\to l} + \dots\right)\otimes\left(\delta_{jm}\,\delta(1-x_2) + \Gamma^{(1)}_{j\to m} + \Gamma^{(2)}_{j\to m} + \dots\right)$$

$$\otimes\left(d\hat{G}^{(0)}_{lm} + d\hat{G}^{(1)}_{lm} + d\hat{G}^{(2)}_{lm} + \dots\right)$$

$$= d\hat{G}^{(0)}_{ij} +$$

$$+ d\hat{G}^{(1)}_{ij} + \Gamma^{(1)}_{j\to m}\otimes d\hat{G}^{(0)}_{im} + \Gamma^{(1)}_{i\to l}\otimes d\hat{G}^{(0)}_{lj}$$

$$+ d\hat{G}^{(2)}_{ij} + \Gamma^{(1)}_{j\to m}\otimes d\hat{G}^{(1)}_{im} + \Gamma^{(1)}_{i\to l}\otimes d\hat{G}^{(1)}_{lj} +$$

$$+ \Gamma^{(1)}_{i\to l}\otimes\Gamma^{(1)}_{j\to m}\otimes d\hat{G}^{(0)}_{lm} + \Gamma^{(2)}_{j\to m}\otimes d\hat{G}^{(0)}_{im} + \Gamma^{(2)}_{i\to l}\otimes d\hat{G}^{(0)}_{lj}$$

$$\Rightarrow \quad \underline{\underline{d\hat{G}^{(0)}_{ij} = d\tilde{G}^{(0)}_{ij}}}$$

$$\underline{\underline{d\hat{G}^{(1)}_{ij} = d\tilde{G}^{(1)}_{ij} - \Gamma^{(1)}_{i\to\ell} \otimes d\tilde{G}^{(0)}_{\ell j} - \Gamma^{(1)}_{j\to m} \otimes d\tilde{G}^{(0)}_{im}}}$$

$$\begin{aligned}
d\hat{G}^{(2)}_{ij} = \; & d\tilde{G}^{(2)}_{ij} - \Gamma^{(1)}_{i\to\ell} \otimes d\hat{G}^{(1)}_{\ell j} - \Gamma^{(1)}_{j\to m} \otimes d\hat{G}^{(1)}_{im} \\
& - \Gamma^{(1)}_{i\to\ell} \otimes \Gamma^{(1)}_{j\to m} \otimes d\tilde{G}^{(0)}_{\ell m} - \Gamma^{(2)}_{j\to m} \otimes d\tilde{G}^{(0)}_{im} - \Gamma^{(2)}_{i\to\ell} \otimes d\tilde{G}^{(0)}_{\ell j}
\end{aligned}$$

$$\begin{aligned}
= \; & d\tilde{G}^{(2)}_{ij} - \Gamma^{(1)}_{i\to\ell} \otimes \Big[d\tilde{G}^{(1)}_{\ell j} - \Gamma^{(1)}_{\ell\to\ell'} \otimes d\tilde{G}^{(0)}_{\ell' j} - \cancel{\Gamma^{(1)}_{j\to m'} \otimes d\tilde{G}^{(0)}_{\ell m'}} \Big] \\
& - \Gamma^{(1)}_{j\to m} \otimes \Big[d\tilde{G}^{(1)}_{im} - \Gamma^{(1)}_{i\to\ell'} \otimes d\tilde{G}^{(0)}_{\ell' m} - \Gamma^{(1)}_{m\to m'} \otimes d\tilde{G}^{(0)}_{im'} \Big] \\
& - \cancel{\Gamma^{(1)}_{i\to\ell} \otimes \Gamma^{(1)}_{j\to m} \otimes} d\tilde{G}^{(0)}_{\ell m} - \Gamma^{(2)}_{j\to m} \otimes d\tilde{G}^{(0)}_{im} - \Gamma^{(2)}_{i\to\ell} \otimes d\tilde{G}^{(0)}_{\ell j}
\end{aligned}$$

$$\begin{aligned}
= \; & d\tilde{G}^{(2)}_{ij} - \Gamma^{(1)}_{i\to\ell} \otimes d\tilde{G}^{(1)}_{\ell j} - \Gamma^{(1)}_{j\to m} \otimes d\tilde{G}^{(1)}_{im} \\
& + \Gamma^{(1)}_{i\to\ell} \otimes \Gamma^{(1)}_{\ell\to\ell'} \otimes d\tilde{G}^{(0)}_{\ell' j} + \Gamma^{(1)}_{j\to m} \otimes \Gamma^{(1)}_{m\to m'} \otimes d\tilde{G}^{(0)}_{im'} \\
& + \Gamma^{(1)}_{i\to\ell} \otimes \Gamma^{(1)}_{j\to m} \otimes d\tilde{G}^{(0)}_{\ell m} \\
& - \Gamma^{(2)}_{j\to m} \otimes d\tilde{G}^{(0)}_{im} - \Gamma^{(2)}_{i\to\ell} \otimes d\tilde{G}^{(0)}_{\ell j}
\end{aligned}$$

$$\begin{aligned}
= \; & d\tilde{G}^{(2)}_{ij} - \Gamma^{(1)}_{i\to\ell} \otimes d\tilde{G}^{(1)}_{\ell j} - \Gamma^{(1)}_{j\to m} \otimes d\tilde{G}^{(1)}_{im} \\
& + \left(\Gamma^{(1)}_{i\to\ell'} \otimes \Gamma^{(1)}_{\ell'\to\ell} - \Gamma^{(2)}_{i\to\ell'} \right) \otimes d\tilde{G}^{(0)}_{\ell' j} \\
& + \left(\Gamma^{(1)}_{j\to m'} \otimes \Gamma^{(1)}_{m'\to m} - \Gamma^{(2)}_{j\to m} \right) \otimes d\tilde{G}^{(0)}_{im} \\
& + \Gamma^{(1)}_{i\to\ell} \otimes \Gamma^{(1)}_{j\to m} \otimes d\tilde{G}^{(0)}_{\ell m}
\end{aligned}$$

d)

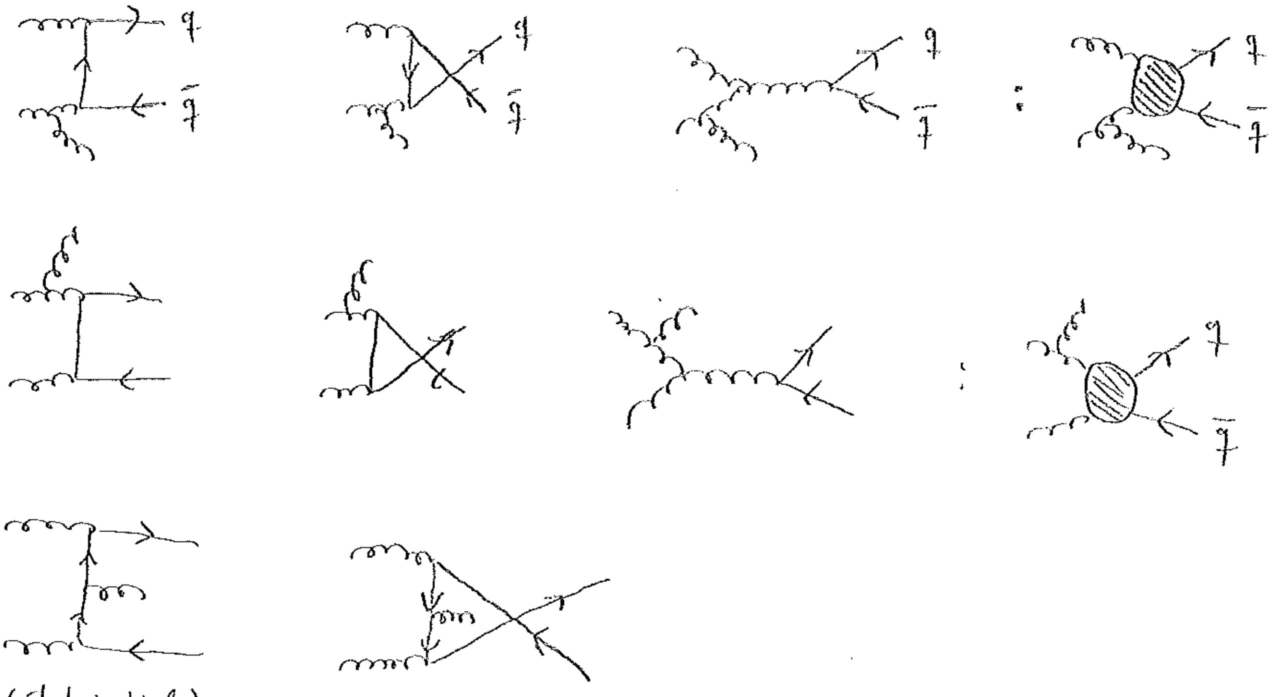
$$g + g \to q + \bar{q} + g$$

$$d\hat{\sigma}^{(1)}_{gg\to q\bar{q}g} = d\tilde{\sigma}^{(1)}_{gg\to q\bar{q}g} - \Gamma^{(1)}_{g\to g}(x_1) \otimes d\tilde{\sigma}^{(0)}_{gg\to q\bar{q}} - \Gamma^{(1)}_{g\to g}(x_2) \otimes d\tilde{\sigma}^{(0)}_{gg\to q\bar{q}}$$

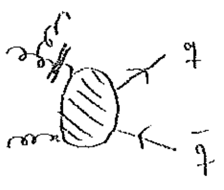
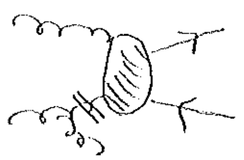
$$- \Gamma^{(1)}_{g\to q}(x_1) \otimes d\tilde{\sigma}^{(0)}_{qg\to gq} - \Gamma^{(1)}_{g\to \bar{q}}(x_1) \otimes d\tilde{\sigma}^{(0)}_{\bar{q}g\to g\bar{q}}$$

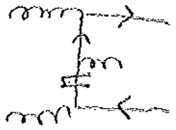
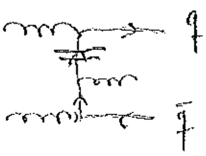
$$- \Gamma^{(1)}_{g\to q}(x_2) \otimes d\tilde{\sigma}^{(0)}_{gq\to qg} - \Gamma^{(1)}_{g\to \bar{q}}(x_2) \otimes d\tilde{\sigma}^{(0)}_{g\bar{q}\to \bar{q}g}$$

<u>Diagrammes:</u>



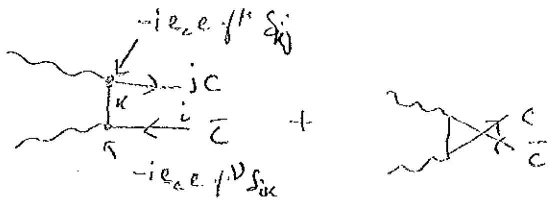
<u>Div. C:</u> (état initial)

 $\hat{=}\ \Gamma^{(1)}_{g\to g}(x_1) \otimes d\tilde{\sigma}^{(0)}_{gg\to q\bar{q}}$;  $\hat{=}\ \Gamma^{(1)}_{g\to g}(x_2) \otimes d\tilde{\sigma}^{(0)}_{gg\to q\bar{q}}$

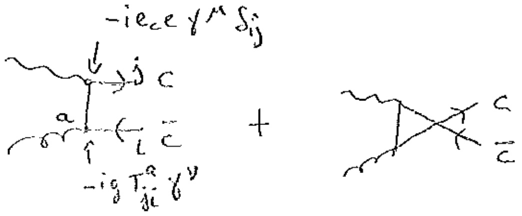
 $\hat{=}\ \Gamma^{(1)}_{g\to q}(x_2) \otimes d\tilde{\sigma}^{(0)}_{gq\to qg}$;  $\hat{=}\ \Gamma^{(1)}_{g\to \bar{q}}(x_1) \otimes d\tilde{\sigma}^{(0)}_{\bar{q}g\to g\bar{q}}$

2)

$\gamma + \gamma \to c + \bar{c}$:



$\gamma + g \to c + \bar{c}$



$$G(\gamma\gamma \to c\bar{c}) \sim (e_c^2 e^2)^2 \sum_{i,j=1}^{N_c} \delta_{ij} = e^4 e_c^4 N_c$$

$$G(\gamma g \to c\bar{c}) \sim (e e_c g)^2 \cdot \frac{1}{N_c^2-1} \cdot \sum_a \underbrace{\sum_{ij} T^a_{ji} T^a_{ij}}_{\mathrm{Tr}(T^a T^a) = \frac{1}{2}\delta^{aa} = T_R \delta^{aa}}$$

$$= e^2 e_c^2 g^2 \underbrace{\frac{1}{N_c^2-1} \frac{1}{2} \sum_{a=1}^{N_c^2-1} \delta^{aa}}_{= \frac{1}{2} = T_R}$$

$$\frac{G(\gamma g \to c\bar{c})}{G(\gamma\gamma \to c\bar{c})} = \frac{\frac{1}{2} e^2 e_c^2 g^2}{e^4 e_c^4 N_c} = \frac{1}{2} \frac{g^2}{e^2 e_c^2 N_c} = \frac{1}{2N_c} \cdot \frac{\alpha_s}{\alpha} \frac{1}{e_c^2} = \boxed{\frac{T_R}{N_c} \frac{1}{e_c^2} \frac{\alpha_s}{\alpha}}$$

$$= \boxed{\frac{1}{6} \frac{9}{4} \frac{\alpha_s}{\alpha} = \frac{3}{8} \frac{\alpha_s}{\alpha}}$$

$$= \frac{1}{6 e_c^2} \frac{g_s^2}{e^2} = 5{,}1 \quad \text{avec } \alpha_s = 0{,}1,\ \alpha = \frac{1}{137}$$

$$= 10{,}2 \quad \text{avec } \alpha_s = 0{,}2,\ \alpha = \frac{1}{137}$$

3)

a)

$$\mathcal{L}^0_G = -\frac{1}{4}(\partial_\mu G_\nu^{\ a} - \partial_\nu G_\mu^{\ a})(\partial^\mu G^\nu_{\ a} - \partial^\nu G^\mu_{\ a})$$

$$= \frac{1}{2} G^\mu_{\ a} \left[\Box g_{\mu\nu} - \partial_\mu \partial_\nu \right] G^\nu_{\ b}\, \delta_{ab} \qquad ; \quad \mathcal{L}_{GF} = -\frac{1}{2\xi}(\partial^\mu G_\mu^{\ a})^2$$

$$\mathcal{L}^0_{G+GF} = \frac{1}{2} G^\mu_{\ a} \left[\Box g_{\mu\nu} - \left(1-\frac{1}{\xi}\right) \partial_\mu \partial_\nu \right] G^\nu_{\ b}\, \delta_{ab}$$

$$\Rightarrow \quad V_{GG} = i\left(-p^2 g_{\mu\nu} + \left(1-\frac{1}{\xi}\right) p_\mu p_\nu \right) \delta_{ab}$$

$$i\tilde{D}^{\mu\nu}_{ab}(p) = \frac{i}{p^2}\left[-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2} \right] \delta_{ab}$$

b)

$$\mathcal{L}_{GF} = -\frac{1}{2\xi}(b^\mu G_\mu^{\ a})(b^\nu G_\nu^{\ a})$$

$$\Rightarrow \quad \mathcal{L}^0_{G+GF} = \frac{1}{2} G_\mu^{\ a} \left[\Box g_{\mu\nu} - \partial_\mu \partial_\nu - \frac{1}{\xi} b_\mu b_\nu \right] G_\nu^{\ b}\, \delta_{ab}$$

1. $i\mathcal{L}$

2. $\rightarrow \quad i\,\frac{1}{2} G_\mu^{\ a} \left[-p^2 g_{\mu\nu} + p_\mu p_\nu - \frac{1}{\xi} b_\mu b_\nu \right] G_\nu^{\ b}\, \delta_{ab}$

3. $\rightarrow$ Facteur 2! (symétrisation)

4. $$\underline{\underline{V_{GG} = i\left[-p^2 g_{\mu\nu} + p_\mu p_\nu - \frac{1}{\xi} b_\mu b_\nu \right] \delta_{ab}}}$$

[Test : dans la limite $b_\mu = p_\mu$ on retrouve la jauge covariante.]

Prop. :

$$V_{GG} = -\left(P_{AA}^{-1}\right)_{\mu\nu}$$

Ansatz : $$P_{AA\,\mu\nu} \equiv P_{\mu\nu} = A\, g_{\mu\nu} + B\, p_\mu p_\nu + C\, b_\mu b_\nu + D(p_\mu b_\nu + p_\nu b_\mu)$$

$$P_{\mu\nu} V^{\nu\rho}_{GG} \stackrel{!}{=} -\delta_\mu^\rho$$

$$V_{GG}{}^{\nu\rho} = i\left[-p^2 g^{\nu\rho} + p^\nu p^\rho - \frac{1}{\xi} b^\nu b^\rho\right]\delta_{ab}$$

$A:$ $$g_{\mu\nu} V^{\nu\rho} = i\left[-p^2\delta_\mu^\rho + p_\mu p^\rho - \frac{1}{\xi} b_\mu b^\rho\right]\delta_{ab}$$

$B:$ $$p_\mu p_\nu V^{\nu\rho} = i\left[-p^2 p_\mu p^\rho + p^2 p_\mu p^\rho - \frac{1}{\xi} p\cdot b\; p_\mu b^\rho\right]\delta_{ab}$$
$$= i\left[-\frac{1}{\xi} p\cdot b\; p_\mu b^\rho\right]\delta_{ab}$$

$C:$ $$b_\mu b_\nu V^{\nu\rho} = i\left[-p^2 b_\mu b^\rho + b_\mu p^\rho\, b\cdot p - \frac{1}{\xi} b_\mu b^\rho\, b^2\right]\delta_{ab}$$
$$= i\left[b_\mu b^\rho\left(-p^2 - \frac{1}{\xi} b^2\right) + b_\mu p^\rho\, b\cdot p\right]\delta_{ab}$$

$D:$ $$(p_\mu b_\nu + p_\nu b_\mu) V^{\nu\rho} = i\Big[-p^2 p_\mu b^\rho + p_\mu p^\rho\, b\cdot p - \frac{1}{\xi} p_\mu b^\rho\, b^2$$
$$-\cancel{p^2 b_\mu p^\rho} + \cancel{p^2 b_\mu p^\rho} - \frac{1}{\xi} b_\mu b^\rho\, p\cdot b\Big]\delta_{ab}$$
$$= i\left[p_\mu b^\rho\left(-p^2 - \frac{1}{\xi} b^2\right) + p_\mu p^\rho\, b\cdot p + b_\mu b^\rho\left(-\frac{1}{\xi} p\cdot b\right)\right]\delta_{ab}$$

Supprimer le facteur δ_{ab} :

$$\Rightarrow \quad -i p^2 A = -1 \quad \Rightarrow \quad \boxed{A = \frac{-i}{p^2}}$$

$$p_\mu p^\rho\left(A + b\cdot p\, D\right) = 0 \quad \Rightarrow \quad \boxed{D = \frac{-A}{b\cdot p} = \frac{i}{p^2}\frac{1}{b\cdot p}}$$

$$b_\mu b^\rho\left(-\frac{1}{\xi} A - C\left(p^2 + \frac{1}{\xi} b^2\right) - D\frac{1}{\xi} p\cdot b\right) = 0 \quad \Rightarrow \quad \boxed{C = 0}$$

$$p_\mu b^\rho\left(-\frac{1}{\xi} p\cdot b\, B - D\left(p^2 + \frac{1}{\xi} b^2\right)\right) = 0 \quad \Rightarrow B = -\frac{D\left(p^2 + \frac{1}{\xi} b^2\right)\xi}{p\cdot b}$$

$$\Rightarrow \boxed{B = -\frac{i}{p^2}\frac{1}{(p\cdot b)^2}\left(b^2 + \xi p^2\right)}$$

$$b_\mu p^\rho\left(C\, b\cdot p\right) = 0 \quad \Rightarrow \quad C = 0 \;\checkmark$$

$$\Rightarrow \quad i\tilde{D}^{ab}_{\mu\nu}(p) = \frac{i}{p^2}\left[-g_{\mu\nu} + \frac{p_\mu b_\nu + p_\nu b_\mu}{b\cdot p} - \frac{(b^2 + \xi p^2)\, p_\mu p_\nu}{(b\cdot p)^2} \right] \delta_{ab}$$

Dans la limite $b_\mu = p_\mu$ on retrouve bien le prop. dans la jauge covariante !

c)

$\xi = 0$, $b^2 = 0$ (jauge de cône de lumière)

$$d_{\mu\nu} = -g_{\mu\nu} + \frac{p_\mu b_\nu + p_\nu b_\mu}{b\cdot p}$$

$$b^\mu d_{\mu\nu} = -b_\nu + \frac{b\cdot p\; b_\nu}{b\cdot p} = 0$$

$$p^\mu d_{\mu\nu} = -p_\nu + \frac{p^2 b_\nu + b\cdot p\, p_\nu}{b\cdot p} \underset{p^2\to 0}{\longrightarrow} 0$$

d)

$\Rightarrow$ Uniquement les polarisations transversales (physiques) propagent

$\Rightarrow$ $\mathcal{L}_{FP} = 0$