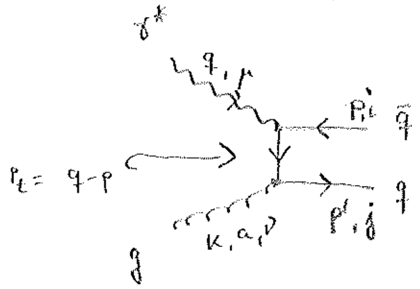
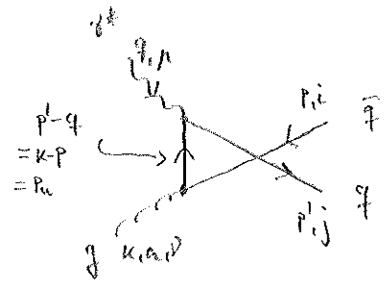
Ex. 1 : Calcul de $P_{qg}(x)$

$$\gamma^*(q) + g(k) \longrightarrow \bar q(p) + q(p') \quad , \; s=(q+k)^2 , \; t=(q-p)^2 , \; u=(q-p')^2$$

a) M_1 $\qquad\qquad$ M_2





$i, j = 1, 2, 3$
$a = 1, 2, \dots, 8$ $\Big\}$ indices de couleur

μ : indice de Lorentz du photon
ν : " " " du gluon

b) "Voie t" :

$$-iM_1 = \bar u(p')\left(-ig\gamma^\nu T^a_{ji}\right)\frac{i}{\not q - \not p - m}\left(-iee_q\gamma^\mu\right)v(p)\;\epsilon^a_\nu(k)\,\epsilon_\mu(q)$$

où $e_q = \frac{2}{3}$ si $q \in \{u, c, t\}$ et $e_q = -\frac{1}{3}$ si $q \in \{d, s, b\}$

$$\Rightarrow \quad M_1 = e\,e_q\,g\,\bar u(p')\left[\gamma^\nu \frac{\not q - \not p + m}{(q-p)^2 - m^2}\gamma^\mu\right]v(p)\;T^a_{ji}\,\epsilon^a_\nu(k)\,\epsilon_\mu(q)$$

"Voie u" :

$$-iM_2 = \bar u(p')\left(-iee_q\gamma^\mu\right)\frac{i}{\not p' - \not q - m}\left(-ig\gamma^\nu T^a_{ji}\right)v(p)\;\epsilon^a_\nu(k)\,\epsilon_\mu(q)$$

$$\Rightarrow \quad M_2 = e\,e_q\,g\,\bar u(p')\left[\gamma^\mu \frac{\not p' - \not q + m}{(p'-q)^2 - m^2}\gamma^\nu\right]v(p)\;T^a_{ji}\,\epsilon^a_\nu(k)\,\epsilon_\mu(q)$$

c) Facteur de couleur dans $\overline{|M|^2}$

$$\frac{1}{8}\sum_{a=1}^{8}\sum_{i,j=1}^{3} T^a_{ji}\left(T^a_{ji}\right)^* = \frac{1}{8}\sum_{a=1}^{8}\sum_{i,j=1}^{3} T^a_{ji}\,T^a_{ij} \qquad ; \; T^a = T^{a\dagger} \Rightarrow (T^a)^* = (T^a)^T$$

$$= \frac{1}{8}\sum_a \underbrace{\mathrm{Tr}\left(T^a T^a\right)}_{= T_R\,\delta^{aa} = \frac{1}{2}\delta^{aa}} = T_R = \frac{1}{2}$$

d)

Dans le cours, pour $\gamma^* + q \longrightarrow g + q$, on avait :

$$\overline{|M|}^2\Big|_{\epsilon_\mu\epsilon_\nu^* = -g_{\mu\nu}} = -C_F\, e_q^2\, 4\, g_s^2 \left(\frac{u}{s} + \frac{s}{u} + \frac{2tq^2}{su} \right)$$

où : C_F était le facteur de couleur

un facteur e^2 a été supprimé

Pour $\gamma^* + g \longrightarrow \bar{q} + q$ symétrie de croisement :

* $s \leftrightarrow t$, un signe (-1) pour chaque fermion croisé entre état initial et état final
* $C_F \longrightarrow T_R = \frac{1}{2}$ (voir Peskin & Schroeder, p. 155)

Donc :

$$\overline{|M|}^2\Big|_{\epsilon_\mu\epsilon_\nu^* = -g_{\mu\nu}} = +T_R\, e_q^2\, 4\, g_s^2 \left(\frac{u}{t} + \frac{t}{u} + \frac{2sq^2}{tu} \right)$$

e)

On a (voir cours) :

$$\frac{1}{x} F_2 \overset{LLA}{\simeq} -g^{\mu\nu} W_{\mu\nu} = \frac{1}{4\pi} \int \overline{|M|}^2\Big|_{\epsilon_\mu\epsilon_\nu^* = -g_{\mu\nu}} dQ_2$$ (leading log approximation)

où F_2, $W_{\mu\nu}$ designe la fonction de structure, le tenseur au niveau partonique

et dQ_2 l'espace de phase

$$dQ_2 = \frac{d^3 p_q}{2p_q^0 (2\pi)^3} \frac{d^3 p_{\bar{q}}}{2p_{\bar{q}}^0 (2\pi)^3} (2\pi)^4 \delta^{(4)}(q + k - p_q - p_{\bar{q}})$$

$$= \frac{1}{32\pi^2} d\Omega^* = \frac{1}{8\pi} \frac{1}{s - q^2} du = \frac{1}{8\pi} \frac{1}{s-q^2} dt$$

De plus : $s + t + u = q^2$, $u_{min} = -(s - q^2)$; $Q^2 := -q^2 > 0$

$u_{max} = 0$

$$\Rightarrow \quad \frac{F_2}{x} \overset{LLA}{\simeq} \frac{1}{32\pi^2} \frac{1}{s+Q^2} T_R e_q^2 \, 4 g_s^2 \int_{-(s+Q^2)+\mu^2}^{-\mu^2} du \left(\frac{u}{t} + \frac{t}{u} - \frac{2sQ^2}{tu} \right)$$

$-\mu^2$ ← IR cut-off (regulariser div. à $u=0$)

$-(s+Q^2)+\mu^2$ ← IR cut-off (regulariser div. à $t=0$)

$$x := \frac{Q^2}{2qK} \quad \text{(Bjorken-x au niveau partonique)}$$

$$s = (q+K)^2 = \frac{Q^2}{x}(1-x) \quad , \quad s+Q^2 = \frac{Q^2}{x} =: s_1$$

$$s+t+u = -Q^2 \quad \Rightarrow \quad t = -(s+Q^2) - u \quad = -(s_1+u)$$

$$\Rightarrow \quad \frac{u}{t} = -\frac{u}{u+s_1} = -\frac{u+s_1-s_1}{u+s_1} = -\left(1 - \frac{s_1}{u+s_1} \right) \overset{LLA}{\simeq} \frac{s_1}{u+s_1}$$

1 → Seulement contribution non-log.

$$\frac{t}{u} = -\frac{s_1+u}{u} = -\left(1+\frac{s_1}{u}\right) \overset{LLA}{\simeq} -\frac{s_1}{u}$$

$$\frac{1}{tu} = -\frac{1}{u(s_1+u)} \overset{!}{=} \frac{A}{u} + \frac{B}{s_1+u} = \frac{A(s_1+u)+Bu}{u(s_1+u)} \Rightarrow \left.\begin{array}{l} A s_1 = -1 \\ A+B=0 \end{array}\right\} \Rightarrow \begin{array}{l} A = -\frac{1}{s_1} \\ B = \frac{1}{s_1} \end{array}$$

$$= \frac{1}{s_1}\left(\frac{1}{s_1+u} - \frac{1}{u} \right)$$

$$\Rightarrow \int_{-s_1+\mu^2}^{-\mu^2} du \left[\frac{u}{t} + \frac{t}{u} - \frac{2sQ^2}{tu} \right] \overset{LLA}{\simeq} \overbrace{s_1 \ln\left|\frac{-\mu^2+s_1}{\mu^2}\right|}^{\overset{LLA}{\simeq} \ln\frac{s_1}{\mu^2}} - \overbrace{s_1 \ln\left|\frac{-\mu^2}{-s_1+\mu^2}\right|}^{\overset{LLA}{\simeq} -\ln\frac{s_1}{\mu^2}}$$

$$- 2sQ^2 \frac{1}{s_1} \left(\underbrace{\ln\left|\frac{-\mu^2+s_1}{\mu^2}\right|}_{\ln\frac{s_1}{\mu^2}} - \underbrace{\ln\left|\frac{-\mu^2}{-s_1+\mu^2}\right|}_{-\ln\frac{s_1}{\mu^2}} \right)$$

$$\overset{LLA}{\simeq} 2 s_1 \ln\frac{s_1}{\mu^2} - 4sQ^2 \frac{1}{s_1} \ln\frac{s_1}{\mu^2}$$

$$= s_1 \ln\frac{s_1}{\mu^2} \left(2 - 4x(1-x) \right)$$

; $s = s_1(1-x)$

$Q^2 = s_1 x$

$$\ln\frac{s_1}{\mu^2} = \ln\frac{Q^2}{x\mu^2} \overset{LLA}{\simeq} \ln\frac{Q^2}{\mu^2}$$

$$\Rightarrow \frac{F_2}{x} \overset{LLA}{\simeq} e_q^2 \frac{1}{8\pi^2}(4\pi\alpha_s) T_R \cancel{\frac{1}{s_1}} \cancel{s_1} (2-4x(1-x)) \ln\frac{Q^2}{\mu^2}$$

$$= e_q^2 \frac{\alpha_s}{2\pi} 2 \underbrace{T_R \left(1-2x(1-x)\right)}_{= P_{qg}(x)} \ln\frac{Q^2}{\mu^2} \overset{!}{=} e_q^2 \frac{\alpha_s}{2\pi} 2 P_{qg}(x) \ln\frac{Q^2}{\mu^2}$$

t-channel and u-channel pole: $P_{qg}+P_{\bar{q}g} = 2P_{qg}$

$$\Rightarrow \quad P_{qg}(x) = T_R\left(1-2x(1-x)\right) = T_R\left[x^2+(1-x)^2\right] \quad , \; T_R = \frac{1}{2}$$

Ex. 2 : Asymptotic momentum fractions

a)

Équations d'évolution pour les quarks dans l'espace de Mellin :

$$\dot{q}_i = P_{q_i q_j} q_j + P_{q_i \bar{q}_j} \bar{q}_j + P_{q_i g}\, g$$

$$\dot{\bar{q}}_i = P_{\bar{q}_i q_j} q_j + P_{\bar{q}_i \bar{q}_j} \bar{q}_j + P_{\bar{q}_i g}\, g$$

Ici : $\dot{q}_i \equiv t \frac{\partial}{\partial t} q_i$, on somme sur j (convention d'Einstein)

On définit : $q_i^+ := q_i + \bar{q}_i$, $\bar{q}_i^+ = q_i^+$

$$\Sigma := \sum_{i=1}^{n_f} q_i^+ \quad \text{(singulet)}$$

$P_{q_i g} = P_{\bar{q}_i g} = P_{qg}$, $P_{q_i q_j} = P_{\bar{q}_i \bar{q}_j}$, $P_{q_i \bar{q}_j} = P_{\bar{q}_i q_j}$ (charge conjugation symmetry)

$$\Rightarrow \quad \dot{q}_i^+ = P_{q_i q_j} q_j^+ + P_{q_i \bar{q}_j} \bar{q}_j^+ + 2 P_{q_i g}\, g$$

$$= (\delta_{ij} P^V_{qq} + P^S_{qq}) q_j^+ + (\delta_{ij} P^V_{q\bar{q}} + P^S_{q\bar{q}}) \bar{q}_j^+ + 2 P_{qg}\, g$$

$$= (P^V_{qq} + P^V_{q\bar{q}}) q_i^+ + (P^S_{qq} + P^S_{q\bar{q}}) \Sigma + 2 P_{qg}\, g$$

$\Sigma = \sum_{i=1}^{n_f} q_i^+$
$\Rightarrow$

$$\dot{\Sigma} = (P^V_{qq} + P^V_{q\bar{q}}) \Sigma + n_f (P^S_{qq} + P^S_{q\bar{q}}) \Sigma + 2 n_f P_{qg}\, g$$

$$\overset{!}{=} P_{qq} \Sigma + 2 n_f P_{qg}\, g$$

$$\Rightarrow \quad \underline{\underline{P_{qq} = P^V_{qq} + P^V_{q\bar{q}} + n_f (P^S_{qq} + P^S_{q\bar{q}})}}$$

À leading order : $P^{(0)}_{qq} = P^{V(0)}_{qq}$

Alternativement :

$$P_{qq} = \frac{1}{2 n_f} \Big[\sum_{i,j} (P_{q_i q_j} + P_{q_i \bar{q}_j}) + (P_{\bar{q}_i q_j} + P_{\bar{q}_i \bar{q}_j}) \Big]$$

b)

$$t \frac{\partial}{\partial t} \begin{pmatrix} \Sigma(N,t) \\ G(N,t) \end{pmatrix} = \frac{\alpha_s(t)}{2\pi} \begin{pmatrix} P_{qq}(N) & 2 n_f \, P_{qg}(N) \\ P_{gq}(N) & P_{gg}(N) \end{pmatrix} \begin{pmatrix} \Sigma(N,t) \\ G(N,t) \end{pmatrix}$$

Leading order:

$$P_{qq}(x) = C_F \left(\frac{1+x^2}{1-x} \right)_+$$

$$P_{qg}(x) = T_R \left[x^2 + (1-x)^2 \right] \qquad , \; T_R = \frac{1}{2}$$

$$P_{gq}(x) = C_F \; \frac{1+(1-x)^2}{x}$$

$$P_{gg}(x) = 6 \left\{ \left(\frac{1}{1-x}\right)_+ + \frac{1-x}{x} - 1 + x(1-x) \right\} + \frac{33-2n_f}{6} \; \delta(1-x)$$

$$P_{ij}(N) = \int_0^1 dx \; x^{N-1} \, P_{ij}(x)$$

(TD4)

$$\Rightarrow \quad P_{qq}(N) = C_F \left(-\frac{1}{2} + \frac{1}{N(N+1)} - 2 \sum_{k=2}^{N} \frac{1}{k} \right)$$

$$P_{qg}(N) = T_R \; \frac{2+N+N^2}{N(N+1)(N+2)}$$

$$P_{gq}(N) = C_F \; \frac{2+N+N^2}{N(N^2-1)}$$

$$P_{gg}(N) = 2 C_A \left[-\frac{1}{12} + \frac{1}{N(N-1)} + \frac{1}{(N+1)(N+2)} - \sum_{k=2}^{N} \frac{1}{k} \right] - \frac{2}{3} n_f T_R$$

N=2:

$$P_{qq}(2) = C_F \left(-\frac{1}{2} + \frac{1}{6} - 1 \right) = -\frac{4}{3} C_F$$

$$P_{qg}(2) = T_R \; \frac{8}{2\cdot 3\cdot 4} = \frac{1}{3} T_R \qquad \Rightarrow 2 n_f P_{qg}(N=2) = 2 n_f \frac{1}{3} \overset{1/2}{\overset{\shortparallel}{T_R}} = \frac{n_f}{3}$$

$$P_{gq}(2) = C_F \; \frac{8}{2\cdot 3} = \frac{4}{3} C_F$$

$$P_{gg}(2) = 2 C_A \left[\underbrace{-\frac{1}{12} + \frac{1}{2} + \frac{1}{12} - \frac{1}{2}}_{=0} \right] - \frac{2}{3} n_f T_R = -\frac{1}{3} n_f$$

c)

$$t \frac{\partial}{\partial t} \begin{pmatrix} \Sigma(z,t) \\ G(z,t) \end{pmatrix} = \frac{\alpha_s(t)}{2\pi} \underbrace{\begin{pmatrix} -\frac{4}{3} C_F & \frac{1}{3} n_f \\ \frac{4}{3} C_F & -\frac{1}{3} n_f \end{pmatrix}}_{=: M} \begin{pmatrix} \Sigma(z,t) \\ G(z,t) \end{pmatrix}$$

Valeurs propres

$$\det(M) = r_+ \cdot r_- = 0$$

$$\mathrm{tr}(M) = r_+ + r_- = -\frac{1}{3}(4C_F + n_f) < 0$$

$$\Rightarrow \boxed{\begin{array}{l} r_+ = 0 \\ r_- = -\frac{1}{3}(4C_F + n_f) \end{array}}$$

Alternativement : équation charactéristique

$$\det(M - r\,\mathrm{Id}) = \left(-\frac{4}{3}C_F - r\right)\left(-\frac{1}{3}n_f - r\right) - \frac{4}{3}C_F \frac{1}{3} n_f = 0$$

$$\Leftrightarrow \quad (4C_F + 3r)(n_f + 3r) - 4C_F n_f = 0$$

$$\Leftrightarrow \quad 9r^2 + 3r(4C_F + n_f) = 0 \quad \Leftrightarrow \quad r = 0 \vee r = -\frac{1}{3}(4C_F + n_f)$$

Vecteurs propres

$r_+ = 0$: $\quad M \vec{v}_+ = \vec{0} \quad \Rightarrow 4C_F v_1 = n_f v_2 \quad \Rightarrow \vec{v}_+ \propto \begin{pmatrix} 1 \\ \frac{4C_F}{n_f} \end{pmatrix}$

$\Rightarrow$ normalisé : $\vec{e}_+ = \frac{1}{\sqrt{1 + \frac{4C_F^2}{n_f^2}}} \begin{pmatrix} 1 \\ \frac{4C_F}{n_f} \end{pmatrix} = \frac{1}{\sqrt{4C_F^2 + n_f^2}} \begin{pmatrix} n_f \\ 4C_F \end{pmatrix}$

r_- : $\quad (M - r_-\,\mathrm{Id})\vec{v}_- = \vec{0} \quad \Leftrightarrow \quad \begin{pmatrix} \frac{1}{3} n_f & \frac{1}{3} n_f \\ \frac{4}{3} C_F & \frac{4}{3} C_F \end{pmatrix} \vec{v}_- = \vec{0}$

$\Rightarrow \quad v_2 = -v_1 \quad \Rightarrow \vec{v}_- \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad , \quad \vec{e}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Diagonaliser :

$$\left.\begin{array}{l} \vec{e}_+ = \frac{n_f}{N} \vec{e}_1 + \frac{4C_F}{N} \vec{e}_2 \quad , \quad N = \frac{1}{\sqrt{4C_F^2 + n_f^2}} \\ \vec{e}_- = \frac{1}{\sqrt{2}} \vec{e}_1 - \frac{1}{\sqrt{2}} \vec{e}_2 \end{array}\right\} \Rightarrow \begin{array}{l} \vec{e}_1 = \frac{N}{4C_F + n_f}\left(\vec{e}_+ + \frac{4C_F}{N}\sqrt{2}\,\vec{e}_-\right) \\ \vec{e}_2 = \frac{N}{4C_F + n_f}\left(\vec{e}_+ - \frac{n_f}{N}\sqrt{2}\,\vec{e}_-\right) \end{array}$$

$$\Rightarrow \quad \Sigma \vec{e}_1 + G \vec{e}_2 = \frac{N}{4C_F + n_f}\left[\underbrace{(\Sigma + G)}_{=: O^+} \vec{e}_+ + \frac{4C_F\sqrt{2}}{N} \vec{e}_- \underbrace{\left(\Sigma - \frac{n_f}{4C_F} G\right)}_{=: O^-}\right]$$

$$t\frac{\partial}{\partial t}\left(\Sigma\,\vec{e}_1 + G\,\vec{e}_2\right) = t\frac{\partial}{\partial t}\frac{N}{4C_F+n_f}\left[O^+\,\vec{e}_+ + \frac{4C_F\,\Gamma_2}{N}\,O^-\,\vec{e}_-\right]$$

$$= \frac{\alpha_s}{2\pi}\,M\left(\Sigma\,\vec{e}_1 + G\,\vec{e}_2\right)$$

$$= \frac{\alpha_s}{2\pi}\frac{N}{4C_F+n_f}\left[O^+\underbrace{M\vec{e}_+}_{r_+\vec{e}_+} + \frac{4C_F\,\Gamma_2}{N}\,O^-\underbrace{M\vec{e}_-}_{r_-\vec{e}_-}\right]$$

$$\Rightarrow\quad t\frac{\partial}{\partial t}O^+ = \frac{\alpha_s}{2\pi}\,r_+\,O^+$$

$$t\frac{\partial}{\partial t}O^- = \frac{\alpha_s}{2\pi}\,r_-\,O^-$$

$$\Leftrightarrow\quad t\frac{\partial}{\partial t}\begin{pmatrix}O^+\\O^-\end{pmatrix} = \frac{\alpha_s}{2\pi}\begin{pmatrix}r_+ & 0\\0 & r_-\end{pmatrix}\begin{pmatrix}O^+\\O^-\end{pmatrix}$$

avec : $O^+ = \Sigma + G \quad , \quad r_+ = 0$

$O^- = \Sigma - \frac{n_f}{4C_F}G \quad , \quad r_- = -\frac{1}{3}\left(4C_F + n_f\right)$

Interprétation de $O^+(z,t) = \Sigma(z,t) + G(z,t)$: Impulsion totale portée par les quarks et les gluons

$$\boxed{O^+(z,t) = 1 \quad \forall t}$$

d) Cherche solution de $t \frac{\partial}{\partial t} O^- = \frac{\partial O^-}{\partial \ln t} = \dot{O}^- = \frac{\alpha_s}{2\pi} r_- O^-$

$$\frac{dO^-}{O^-} = \frac{\alpha_s}{2\pi} r_- \, d\ln t \quad ; \quad a_s = \frac{\alpha_s}{4\pi} \, , \quad \frac{da_s}{d\ln t} = -\beta_0 a_s^2$$

$$\Rightarrow d\ln t = \frac{da_s}{-\beta_0 a_s^2}$$

$$\Rightarrow \frac{dO^-}{O^-} = 2 a_s r_- \frac{da_s}{-\beta_0 a_s^2}$$

$$= -\frac{2r_-}{\beta_0} \frac{da_s}{a_s}$$

$$\Rightarrow \ln \frac{O^-(t)}{O^-(t_0)} = -\frac{2r_-}{\beta_0} \ln \frac{a_s(t)}{a_s(t_0)}$$

$$\Rightarrow O^-(t) = \left(\frac{a_s(t)}{a_s(t_0)}\right)^{-\frac{2r_-}{\beta_0}} O^-(t_0) \xrightarrow[t\to\infty]{} 0$$

Car $a_s(t) \xrightarrow[t\to\infty]{} 0$ (liberté asymptotique) , $-\frac{2r_-}{\beta_0} > 0$

$$\Rightarrow \quad \Sigma(2,t) \xrightarrow[t\to\infty]{} \frac{n_f}{4C_F} G(2,t)$$

$$\underline{\underline{\frac{\Sigma(2,t)}{G(2,t)} \xrightarrow[t\to\infty]{} \frac{n_f}{4C_F} = \frac{3}{16} n_f}}$$

Fractions de l'impulsion :

$$\Sigma(2,t) + G(2,t) = 1 \quad \Rightarrow \quad \frac{1}{G(2,t)} = 1 + \frac{\Sigma(2,t)}{G(2,t)} = 1 + \frac{3}{16} n_f = \frac{16+3n_f}{16}$$

$$\Rightarrow G(2,t) = \frac{16}{16+3n_f}$$

$$\underline{\underline{\Sigma(2,t) = 1 - G(2,t) = \frac{3n_f}{16+3n_f}}}$$

(voir le livre de Ellis, Stirling & Webber)