2.1

$V = \{ A \in M_n(\mathbb{C}) \mid A = A^\dagger, \ Tr(A) = 0 \}$ espace vectoriel, $T_a \in V$

$\dim V = ?$

$\dim_{\mathbb{R}}(M_n(\mathbb{C})) = 2n^2$

Contraintes : $A = A^\dagger \Rightarrow A_{ii} \in \mathbb{R} \Rightarrow n$ contraintes (diagonale)

$A_{ij} = A_{ji}^* \quad \text{pour } i > j \Rightarrow (1+2+\dots+n-1)\underset{\text{complexe}}{2} = 2\,\frac{n(n-1)}{2}$ contr.

(Triangle au-dessous de la diagonale)

$Tr(A) = 0 \Rightarrow 1$ Contrainte

Au total : $\dim(V) = 2n^2 - n - n(n-1) - 1 = \underline{\underline{n^2 - 1}} \Rightarrow$ Il y a une base avec n^2-1 éléments

Base de V :

$$(A_{ij}) = \begin{pmatrix} 0 & & 1_{(i,j)} \\ & \ddots & \\ 1_{(j,i)} & & 0 \end{pmatrix} \quad 1 \le i < j \le n \qquad \# = \frac{n(n-1)}{2}$$

$$(B_{ij}) = \begin{pmatrix} 0 & & -i_{(i,j)} \\ & \ddots & \\ +i_{(i,j)} & & 0 \end{pmatrix} \quad 1 \le i < j \le n \qquad \# = \frac{n(n-1)}{2}$$

$$E_i = \mathrm{diag}(0,\dots,0,\underset{i}{1},0,\dots,0,-1) \qquad i = 1,\dots,n-1 \ (\leftrightarrow \text{rang} = n-1)$$

Base : $\{ A_{ij}, B_{ij}, E_i \}$

Pour $n=2$: $A_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$, $B_{12} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2$, $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3$

$$[T_a, T_b]^\dagger = [T_b^\dagger, T_a^\dagger] = [T_b, T_a] = -[T_a, T_b] = -i f_{abc} T_c$$

$$\| $$

$$(i f_{abc})^* T_c^\dagger = -i f_{abc}^* T_c \Rightarrow 0 = -i\underset{\text{somme}}{(f_{abc} - f_{abc}^*) T_c} \Rightarrow \underline{f_{abc} = f_{abc}^* \in \mathbb{R}}$$

T_c lin. indep.

2.2 $\quad Tr(T_a T_b) = T_f \delta_{ab}$, $T_f = 1/2$ $\qquad [T_a, T_b] = i f_{abc} T_c = -[T_b, T_a] = -i f_{bac} T_c$

$[T_a, T_b] = i f_{abc} T_c$ $\qquad \Rightarrow \quad \underline{\underline{f_{abc} = -f_{bac}}}$

$$Tr(T_c [T_a, T_b]) = i f_{abd} \overbrace{Tr(T_c T_d)}^{= T_f \delta_{cd}} = T_f i f_{abd} \delta_{cd} = T_f i f_{abc}$$

$$= Tr(T_c T_a T_b - T_c T_b T_a) = Tr(T_b T_c T_a - T_b T_a T_c)$$

$$= Tr(T_b [T_c, T_a]) = -Tr(T_b [T_a, T_c]) = -T_f i f_{acb}$$

$\Rightarrow \quad \underline{\underline{f_{abc} = -f_{acb}}}$

$\Rightarrow \quad f_{abc}$ entièrement anti-symétrique $\begin{pmatrix} 1) \ f_{abc} = -f_{bac} \\ 2) \ f_{abc} = -f_{acb} \end{pmatrix}$

2.3

a) $H = \{ M \in M_n(\mathbb{C}) \mid M = M^+ \}$

H est un $\mathbb{R}$-espace vectoriel avec base $\{ \mathbb{1}_n, T_1, \dots, T_{n^2-1} \}$

[Rem.: Ce n'est pas un $\mathbb{C}$-espace vectoriel!

Soit $M \in H$: $(aM)^+ = a^* M^+ = a^* M \overset{!}{=} aM \in H \Rightarrow a = a^* \in \mathbb{R}$]

Avec la base:

$$M = m_0 \mathbb{1}_n + m_a T_a \quad , \; m_0, m_a \in \mathbb{R} \; , \; a = 1, \dots, n^2-1$$

$$Tr(M) = m_0 Tr(\mathbb{1}_n) + m_a \underset{\substack{\| \\ 0}}{Tr(T_a)} = m_0 \cdot n \Rightarrow \boxed{m_0 = \frac{1}{n} Tr(M)}$$

$$Tr(M T_a) = m_0 Tr(T_a) + m_b \underbrace{Tr(T_b T_a)}_{= \frac{1}{2}\delta_{ab}} = \frac{1}{2} m_a \Rightarrow \boxed{m_a = 2\, Tr(M \cdot T_a)}$$

Donc:

$$M = \frac{1}{n} Tr(M) \mathbb{1}_n + 2\, Tr(M T_a) T_a$$

$$M_{ij} = \frac{1}{n}(M_{\ell k} \delta_{k\ell}) \delta_{ij} + 2 (M_{\ell k} (T_a)_{k\ell}) (T_a)_{ij}$$

$$= M_{\ell k} \underbrace{\left[\frac{1}{n} \delta_{ij} \delta_{k\ell} + 2 (T_a)_{ij} (T_a)_{k\ell} \right]}_{\delta_{\ell i} \delta_{kj}} \qquad \forall M$$

$$\Rightarrow \quad (T_a)_{ij} (T_a)_{k\ell} = \frac{1}{2} \left(\delta_{i\ell} \delta_{jk} - \frac{1}{n} \delta_{ij} \delta_{k\ell} \right)$$

b)

$$(T^a T^a)_{i\ell} = T^a_{ik} T^a_{k\ell} = \frac{1}{2} \left(\delta_{i\ell} \overset{\substack{n \\ \|}}{\delta_{kk}} - \frac{1}{n} \underbrace{\delta_{ik} \delta_{k\ell}}_{\delta_{i\ell}} \right) = \frac{1}{2}\left(n - \frac{1}{n}\right) \delta_{i\ell} = \frac{n^2-1}{2n} \delta_{i\ell}$$

$$\Rightarrow \quad \underset{\substack{\uparrow \\ \sum_{a=1}^{n^2-1}}}{T^a T^a} = C_F \mathbb{1}_n$$

(Opérateur Casimir dans la représentation fondamentale)

2.4

$$0 = [T^a,\underbrace{[T^b,T^c]}_{i f_{bcd} T^d}] + [T^b,\underbrace{[T^c,T^a]}_{i f_{cad} T^d}] + [T^c,\underbrace{[T^a,T^b]}_{i f_{abd} T^d}]$$

$$= i\{ f_{bcd}[T^a,T^d] + f_{cad}[T^b,T^d] + f_{abd}[T^c,T^d]\}$$

$$= -\{\underbrace{f_{bcd} f_{ade} + f_{cad} f_{bde} + f_{abd} f_{cde}}_{=0 \quad (*)}\} T^e \qquad e \in \{1,\dots,n^2-1\}$$

lin. indep.

$$[X_a, X_b]_{ce} = (X_a)_{cd}(X_b)_{de} - (X_b)_{cd}(X_a)_{de} \qquad \forall c,e$$

$$= f_{bcd} f_{ade} - f_{acd} f_{bde}$$

$$\overset{(*)}{=} - f_{cde} f_{abd} = i f_{abd}(-i f_{dce}) = i f_{abd}(X_d)_{ce}$$

$$\Rightarrow \underline{\underline{[X_a, X_b] = i f_{abc} X_c}}$$

Conclusion:
- X_a et T_a ont les mêmes constantes de structure
- X_a, T_a des représentations différentes de l'algèbre su(3)
- $\exp(i\alpha_a X_a)$, $\exp(i\alpha_a T_a)$ des représentations différentes du groupe SU(3)

Notez:

La déf. $(X_a)_{bc} = -i f_{abc}$ est cohérente avec la déf. (indep. d'une base)

$$ad(t^a)\, t = [t^a, t] \quad \text{pour } t \in su(n)$$

$$ad(t^a)\, t^b = [t^a, t^b] = i f_{abc} t^c = (X^a)_{cb} t^c$$

[Par exemple: $ad(t^1)\, t^2 = i f_{12c} t^c = (X^1)_{12} t^2 + (X^1)_{22} t^2 + (X^1)_{32} t^2 + \dots + (X^1)_{n^2-1,2} t^2 = (X^1)_{c2} t^c$]

$$t^2 \leftrightarrow \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in V_{n^2-1}$$

$$\Rightarrow (X^a)_{cb} = i f_{abc} = -i f_{acb}$$

$$\boxed{\text{Donc} \quad (X^a)_{bc} = -i f_{abc}}$$

2.5

* Donné : $X_a X_a = n \, \mathbb{1}_{n^2-1} =: C_A \, \mathbb{1}_{n^2-1}$ (L'opérateur Casimir dans la représent. adjointe)

$$(X_a X_a)_{bc} = n \, \delta_{bc}$$

$$= (X_a)_{bd} (X_a)_{dc} = (-i)^2 f_{abd} f_{adc}$$

$$= - f_{abd} f_{adc} = f_{bad} f_{cad} = \lambda \, \delta_{bc} \quad \Rightarrow \underline{\underline{\lambda = n = C_A}}$$

* $$\underbrace{T_a T_b T_a}_{= T_a T_b - i f_{abc} T_c} = (T_a T_a) T_b - i f_{abc} T_a T_c = C_F \, T_b + \underbrace{i f_{bac} T_a T_c}_{= i f_{bca} T_c T_a = -i f_{bac} T_c T_a}$$

$$= C_F \, T_b + \frac{i}{2} f_{bac} \underbrace{[T_a, T_c]}_{i f_{acd} T_d}$$

$$= C_F \, T_b - \frac{1}{2} \underbrace{f_{bac} f_{dac}}_{C_A \delta_{bd}} T_d$$

$$= C_F \, T_b - \frac{C_A}{2} \delta_{bd} T_d = \underline{\underline{\left(C_F - \frac{C_A}{2} \right) T_b}}$$

* $$f_{abc} T_a T_b = f_{bac} T_b T_a = - f_{abc} T_b T_a = \frac{1}{2} f_{abc} \underbrace{[T_a, T_b]}_{i f_{abd} T_d}$$

$$= \frac{i}{2} C_A \, \delta_{cd} \, T_d = \underline{\underline{\frac{i}{2} C_A \, T_c}}$$

2.6

$T_a = \frac{1}{2} \sigma_a$, $f_{abc} = \epsilon_{abc}$, $[T_a, T_b] = i\epsilon_{abc} T_c$

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $n = 2$, $C_F = \frac{2^2 - 1}{2 \cdot 2} = \frac{3}{4}$, $C_A = 2$

Ad 2.3a) :

11;11 11;12 12;kl

$$(T_a \otimes T_a)_{ij;kl} = \frac{1}{4} \Bigg\{ \underbrace{\left(\begin{array}{cc|c} 0 & 0 & \sigma_1 \\ 0 & 0 & \\ \hline \sigma_1 & & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{array}\right)}_{\sigma_1 \otimes \sigma_1} + \underbrace{\left(\begin{array}{c|c} 0 & -i\sigma_2 \\ \hline i\sigma_2 & 0 \end{array}\right)}_{\sigma_2 \otimes \sigma_2} + \underbrace{\left(\begin{array}{c|c} \sigma_3 & 0 \\ \hline 0 & -\sigma_3 \end{array}\right)}_{\sigma_3 \otimes \sigma_3} \Bigg\}$$

21;kl 22;kl

$$= \frac{1}{4} \begin{pmatrix} \sigma_3 & \sigma_1 - i\sigma_2 \\ \sigma_1 + i\sigma_2 & -\sigma_3 \end{pmatrix} = \frac{1}{4} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ \hline 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$$\overset{!}{=} \frac{1}{2} (\delta_{il} \delta_{jk})_{ij;kl} - \frac{1}{4} (\mathbb{1}_2 \otimes \mathbb{1}_2)_{ij;kl}$$

$ijkl = 11;11$ 11;12 12;11 12;12

$$= \frac{1}{2} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} \\ 12;22 \\ 12;21 \\ \\ \end{array} - \frac{1}{4} \left(\begin{array}{cc|cc} 1 & & 0 & \\ & 1 & & \\ \hline 0 & & 1 & \\ & & & 1 \end{array}\right) = \frac{1}{4} \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ \hline 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \checkmark$$

21;22 21;22 22;22

b):

$$(T_a)^2 = \frac{1}{4} \{ \underbrace{\sigma_1^2}_{\mathbb{1}_2} + \underbrace{\sigma_2^2}_{\mathbb{1}_2} + \underbrace{\sigma_3^2}_{\mathbb{1}_2} \} = \frac{3}{4} \mathbb{1}_2 = C_F \mathbb{1}_2$$

Ad 2.4 :

$(X_a)_{bc} = -i\epsilon_{abc}$, $a,b,c = 1, \dots, \underbrace{n^2-1}_{=3}$; $X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}$

À montrer : $[X_a, X_b]_{ce} = i\epsilon_{abd}(X_d)_{ce} = \epsilon_{abd}\epsilon_{dce} = \epsilon_{dba}\epsilon_{dec}$ $\qquad X_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$[X_a, X_b]_{ce} = (X_a)_{cd}(X_b)_{de} - (X_b)_{cd}(X_a)_{de}$$

$$= (-i)^2 \epsilon_{acd}\epsilon_{bde} - (-i)^2 \epsilon_{bcd}\epsilon_{ade} = \epsilon_{bcd}\epsilon_{ade} - \epsilon_{acd}\epsilon_{bde}$$

$$= -(\delta_{ba}\delta_{ce} - \delta_{be}\delta_{ca}) + (\delta_{ab}\delta_{ce} - \delta_{ae}\delta_{bc})$$

$$= \delta_{be}\delta_{ca} - \delta_{ae}\delta_{bc} = \underline{\epsilon_{dba}\epsilon_{dec}}$$

Alternatif : calcul explicit avec les matrices 3x3

Ad 2.5

* $\epsilon_{acd}\,\epsilon_{bcd} = \delta_{ab}\,\underbrace{\delta_{cc}}_{=3} - \delta_{ac}\,\delta_{cb} = 3\,\delta_{ab} - \delta_{ab} = 2\,\delta_{ab}$

* $T_a\,T_b\,T_a = (T_a)^2\,T_b \underbrace{- i\,\epsilon_{abc}\,T_a T_c}_{\substack{= i\,\epsilon_{bac}\,T_a T_c \\ = \frac{i}{2}\epsilon_{bac}[T_a, T_c]}} = \frac{3}{4}\,T_b + \frac{i}{2}\,\epsilon_{bac}\,\underbrace{[T_a, T_c]}_{i\,\epsilon_{acd}\,T_d}$

$= \frac{3}{4}\,T_b - \frac{1}{2}\,\underbrace{\epsilon_{bac}\,\epsilon_{dac}}_{2\,\delta_{bd}}\,T_d = (3/4 - 2/2)\,T_b$

* $\epsilon_{ijk}\,T_i\,T_j = \frac{1}{2}\,\epsilon_{ijk}\,[T_i, T_j] = \frac{i}{2}\,\underbrace{\epsilon_{ijk}\,\epsilon_{ijl}}_{2\,\delta_{kl}}\,T_l = \frac{i}{2}\,2\,T_k$