5.1

1. $\mathcal{L}'_{QED} = -\frac{1}{4} F'_{\mu\nu} F'^{\mu\nu} + \bar{\psi}' (i \gamma^\mu D'_\mu - m) \psi' \overset{!}{=} \mathcal{L}_{QED}$

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = F_{\mu\nu} + \underbrace{\partial_\mu \partial_\nu \Lambda - \partial_\nu \partial_\mu \Lambda}_{=0} = F_{\mu\nu} \checkmark$$

$$-m \bar{\psi}' \psi' = -m \bar{\psi} e^{iq\Lambda} e^{-iq\Lambda} \psi = -m \bar{\psi} \psi \checkmark$$

$$\bar{\psi}' i \gamma^\mu D'_\mu \psi' = \bar{\psi} e^{iq\Lambda} i \gamma^\mu (\partial_\mu + iq A'_\mu) e^{-iq\Lambda} \psi$$

$$= \bar{\psi} e^{iq\Lambda} i \gamma^\mu \left(e^{-iq\Lambda} (-iq \partial_\mu \Lambda) + e^{-iq\Lambda} (\partial_\mu + iq A'_\mu) \right) \psi$$

$$= \bar{\psi} i \gamma^\mu (\cancel{-iq \partial_\mu \Lambda} + \underbrace{\partial_\mu + iq A_\mu}_{D_\mu} + \cancel{iq \partial_\mu \Lambda}) \psi$$

$$= \bar{\psi} i \gamma^\mu D_\mu \psi \checkmark$$

$\Rightarrow \mathcal{L}'_{QED} = \mathcal{L}_{QED}$

2.

$$\frac{1}{2} \bar{\psi} i \partial_\mu \gamma^\mu \psi - \frac{1}{2} (\partial_\mu \bar{\psi}) i \gamma^\mu \psi = \frac{1}{2} \bar{\psi} i \partial_\mu \gamma^\mu \psi - \frac{1}{2} \partial_\mu [\bar{\psi} i \gamma^\mu \psi] + \frac{1}{2} \bar{\psi} i \partial_\mu \gamma^\mu \psi$$

$$= \bar{\psi} i \partial_\mu \gamma^\mu \psi - \partial_\mu f^\mu(x)$$

avec $f^\mu(x) = \frac{1}{2} i \bar{\psi} \gamma^\mu \psi$

$\Rightarrow \mathcal{L}'_{QED} = \mathcal{L}_{QED} - \partial_\mu f^\mu(x)$

$\Rightarrow \int_R d^4x\, \mathcal{L}'_{QED} = \int_R d^4x\, \mathcal{L}_{QED}$ si $\psi(x)\big|_{\partial R} = 0$, $\bar{\psi}(x)\big|_{\partial R} = 0$

3.

$$\mathcal{L}'_{QED} = \underbrace{\frac{1}{2} \bar{\psi} i \partial_\mu \gamma^\mu \psi - \frac{1}{2} (\partial_\mu \bar{\psi}) i \gamma^\mu \psi - m \bar{\psi} \psi}_{\mathcal{L}_\psi} + \underbrace{e \bar{\psi} A^\mu \gamma_\mu \psi}_{\mathcal{L}_{\psi A}} \underbrace{- \frac{1}{4} F_{\mu\nu} F^{\mu\nu}}_{\mathcal{L}_A}$$

$$\equiv \mathcal{L}_\psi + \mathcal{L}_{\psi A} + \mathcal{L}_A$$

ψ : champs complex $\Rightarrow$ Variation indépendante de ψ et $\bar{\psi}$ au lieu de $\mathrm{Re}\,\psi$ et $\mathrm{Im}\,\psi$

E.-L.: $\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} - \frac{\partial \mathcal{L}}{\partial \bar{\psi}} = 0$

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = \frac{1}{2} i \cancel{\partial} \psi - m \psi + e \cancel{A} \psi \quad ; \quad \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\psi})} = -\frac{1}{2} i \cancel{\partial} \psi$$

$$\Rightarrow \boxed{(i \cancel{\partial} - m) \psi = -e \cancel{A} \psi}$$

$$\frac{\partial \mathcal{L}}{\partial \psi} = -\frac{1}{2}(\partial_\mu \bar{\psi}) i\gamma^\mu - m\bar{\psi} + e\bar{\psi}\cancel{A}$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} = \frac{1}{2}(\partial_\mu \bar{\psi}) i\gamma^\mu$$

$$\Rightarrow \quad \boxed{i(\partial_\mu \bar{\psi})\gamma^\mu + m\bar{\psi} = e\bar{\psi}\cancel{A}}$$

$$\bar{\psi}(i\overleftarrow{\cancel{\partial}} + m) = e\bar{\psi}\cancel{A}$$

$$\Leftrightarrow \overline{(i\cancel{\partial} - m)\psi} = -\overline{e\cancel{A}\psi}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$\frac{\partial F_{\mu\nu}}{\partial(\partial_\alpha A_\beta)} = g^\alpha_\mu g^\beta_\nu - g^\alpha_\nu g^\beta_\mu$$

$$\Rightarrow \quad \frac{\partial \mathcal{L}_A}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{4}\, 2\,(g^\alpha_\mu g^\beta_\nu - g^\alpha_\nu g^\beta_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$$

$$= -\frac{1}{2}(\partial^\alpha A^\beta - \partial^\beta A^\alpha - \partial^\beta A^\alpha + \partial^\alpha A^\beta) = -F^{\alpha\beta}$$

$$\frac{\partial \mathcal{L}_{A\psi}}{\partial A_\beta} = e\bar{\psi}\gamma^\beta\psi$$

$$\Rightarrow \quad \boxed{+\partial_\alpha F^{\alpha\beta} = -e\bar{\psi}\gamma^\beta\psi} = -e\, j^\beta = j^\beta_{em} = q\, j^\beta \quad ; \quad \boxed{\partial_\alpha F^{\alpha\beta} = q\, j^\beta}$$

$$\Leftrightarrow \quad \boxed{\Box A^\beta - \partial^\beta \underbrace{\partial_\alpha A^\alpha}_{} = q\, j^\beta}$$

$= 0$ avec condition de Lorentz $\partial_\mu A^\mu = 0$

4. $U(1)$: $\quad e^{i\alpha} = \lim\limits_{N\to\infty}\left(1 + \frac{i\alpha}{N}\right)^N$

Considerons les transformations infinitesimales :

$$\psi \to (1 + i\epsilon)\psi$$

$$\bar{\psi} \to (1 - i\epsilon)\bar{\psi}$$

avec ϵ global (c.à.d. ϵ ne dépend pas de x)

Rem. : Ce n'est pas une variation de ψ
parce que $\delta\psi|_{\partial R} \neq 0$

$$0 = \delta\mathcal{L} = \frac{\delta\mathcal{L}}{\delta\psi} i\epsilon\psi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)} \partial_\mu(i\epsilon\psi) + (-i\epsilon\bar{\psi})\frac{\delta\mathcal{L}}{\partial\bar{\psi}} + \partial_\mu(-i\epsilon\bar{\psi})\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}$$

$$\Rightarrow \quad 0 = \epsilon \underbrace{\left[\frac{\delta\mathcal{L}}{\delta\psi} - \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right]}_{=0 \text{ pour des états physiques}} \psi + \epsilon\,\partial_\mu\left[\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\psi\right]$$

$$-\epsilon\bar{\psi}\overbrace{\left[\frac{\delta\mathcal{L}}{\delta\bar{\psi}} - \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right]} - \epsilon\,\partial_\mu\left[\bar{\psi}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}\right] \qquad \forall\epsilon$$

$$\Rightarrow \quad \boxed{0 = \partial_\mu j^\mu_{e.m.}} \quad \text{avec} \quad j^\mu_{e.m.} \sim \underbrace{\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\psi - \bar{\psi}\frac{\delta\mathcal{L}}{\delta(\partial_\mu\bar{\psi})}}_{(*)} = j^\mu$$

Courant de Noether

$$\mathcal{L}'_{QED} \quad \Rightarrow \quad j_\mu = \frac{i}{2}\bar{\psi}\gamma_\mu\psi + \frac{i}{2}\bar{\psi}\gamma_\mu\psi = i\bar{\psi}\gamma_\mu\psi$$

Identification : $j^\mu_{e.m} = q\,\bar{\psi}\gamma^\mu\psi$

Continuité : $\boxed{\partial_\mu j^\mu_{e.m.} = 0}$ $\Rightarrow$ Conservation de charge

$$\frac{\partial}{\partial t} j^0_{e.m.} - \vec{\nabla}\vec{j}_{e.m.} = 0$$

$$q\left[\frac{\partial}{\partial t}\underbrace{\bar{\psi}\gamma^0\psi}_{\substack{=\psi^\dagger\psi \\ =\rho}} - \vec{\nabla}\vec{j}\right] = 0 \qquad \frac{\partial}{\partial t}\int d^3x\,\rho = \int d^3x\,\vec{\nabla}\cdot\vec{j} = 0$$

$$\Rightarrow \quad Q = q\int d^3x\,\rho = \text{cte}$$

(5.2)

$\eta^\mu = (1,0,0,0)$, $\eta\cdot\epsilon_\lambda(k)=0$, $k\cdot\epsilon_\lambda(k)=0$, $\epsilon_\lambda(k)\cdot\epsilon^*_{\lambda'}(k) = -\delta_{\lambda\lambda'}$; $k^2=0$

Sans restriction : $k=(k_0,0,0,k_0)$

a) Polarisation linéaire : $\epsilon_{1,2}(k) = \begin{pmatrix} 0 \\ \vec{\epsilon}_{1,2}(k) \end{pmatrix}$ avec $\vec{\epsilon}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{\epsilon}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

b) Polarisation circulaire : $\vec{\epsilon}_{R,L} = \mp \frac{1}{\sqrt{2}}(\vec{\epsilon}_1 \pm i\vec{\epsilon}_2)$

$$\Rightarrow \quad \vec{\epsilon}_R = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} , \quad \vec{\epsilon}_L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix}$$

$$R^{\mu\nu} := \sum_{\lambda=1,2} \epsilon^\mu_\lambda(k)\,\epsilon^{\nu *}_\lambda(k)$$

* $R^{\mu\nu}$ réel : évident pour pol. linéaire

pour pol. circulaire avec $\vec{\epsilon}_L\cdot\vec{\epsilon}_L^{\,*} + \vec{\epsilon}_R\cdot\vec{\epsilon}_R^{\,*} = \frac{1}{2}\left[(\vec{\epsilon}_1 - i\vec{\epsilon}_2)(\vec{\epsilon}_1 - i\vec{\epsilon}_2)^* + (\vec{\epsilon}_1 + i\vec{\epsilon}_2)(\ldots)^*\right]$

$= \vec{\epsilon}_1\cdot\vec{\epsilon}_1 + \vec{\epsilon}_2\cdot\vec{\epsilon}_2$

* $R^{\mu\nu}$ réel $\Rightarrow R^{\mu\nu} = R^{\nu\mu}$

Ansatz : $R^{\mu\nu} = A\,g^{\mu\nu} + B\,k^\mu k^\nu + C\,\eta^\mu\eta^\nu + D(k^\mu\eta^\nu + k^\nu\eta^\mu)$

I) $k_\mu k_\nu R^{\mu\nu} \underset{\substack{\uparrow \\ k\cdot\epsilon_\lambda(k)=0}}{=} 0 = C\,k_0^2 \Rightarrow C=0$

II) $k_\mu \eta_\nu R^{\mu\nu} = 0 = A\,k_0 + D\,k_0^2 \Rightarrow D = -\frac{A}{k_0}$

III) $g_{\mu\nu}R^{\mu\nu} = -2 = 4A + D\,2k_0 = 4A - 2A = 2A \Rightarrow A = -1$

IV) $\eta_\mu\eta_\nu R^{\mu\nu} = 0 = A + B\,k_0^2 + D\,2k_0 = -A + B\,k_0^2 \Rightarrow B = \frac{-1}{k_0^2}$

Avec $k_0^2 \underset{\substack{\uparrow \\ k^2=0}}{=} |\vec{k}|^2$: $\boxed{R^{\mu\nu} = -g^{\mu\nu} - \frac{1}{|\vec{k}|^2}k^\mu k^\nu + \frac{k^0}{|\vec{k}|^2}(k^\mu\eta^\nu + k^\nu\eta^\mu)}$

Rem. : Ansatz le plus général pour $R^{\mu\nu}$:

$$R^{\mu\nu} = A\,g^{\mu\nu} + B\,k^\mu k^\nu + C\,\eta^\mu\eta^\nu + D\,k^\mu\eta^\nu + E\,k^\nu\eta^\mu + F\,\epsilon^{\mu\nu\rho\sigma}k_\rho\eta_\sigma$$

Analyse plus compliqué mais on trouve le même resultat, notamment $F=0$

5.3

$$A_\mu(x) = \int \frac{d^4y}{(2\pi)^4} D_{\mu\nu}(x-y)\, j^\nu(y) \quad \text{avec} \quad D_{\mu\nu}(z) = g_{\mu\nu} \int d^4q \, e^{iqz} \frac{1}{-q^2}$$

$$\Box_x A_\mu(x) = \int \frac{d^4y}{(2\pi)^4} \left(\Box D_{\mu\nu}(x-y)\right) j^\nu(y)$$

$$\Box D_{\mu\nu}(x-y) = g_{\mu\nu} \int d^4q \, \underbrace{(iq)^2}_{-q^2} e^{iq.(x-y)} \frac{1}{-q^2} = g_{\mu\nu} \int d^4q \, e^{iq(x-y)}$$

$$= g_{\mu\nu} (2\pi)^4 \delta^{(4)}(x-y)$$

$$\Rightarrow \Box A_\mu(x) = \int \frac{d^4y}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(x-y)\, j^\nu(y)\, g_{\mu\nu} = j_\mu(x)$$

$$\Box = \partial_\mu \partial^\mu = -\underbrace{(i\partial_\mu)}_{\hat{p}_\mu}\underbrace{(i\partial^\mu)}_{\hat{p}^\mu} = -\hat{p}^2$$

$$\Box f(x) = \Box \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{f}(k) = \int \frac{d^4k}{(2\pi)^4} (-k)^2 \tilde{f}(k)\, e^{-ikx}$$

$$\Rightarrow \quad \Box \tilde{f}(k) = -k^2 \tilde{f}(k)$$

Ex. 5.4

1. $$(\gamma_5)^2 = -\gamma^0\gamma^1\gamma^2\overbrace{\gamma^3\gamma^0}^{=-\gamma^0\gamma^3}\gamma^1\gamma^2\gamma^3$$
$$= \gamma^0\gamma^1\gamma^2\gamma^0\gamma^3\gamma^1\gamma^2\gamma^3 = \ldots = -\underbrace{(\gamma^0)^2}_{\substack{=g_{00}\\=+1}}\underbrace{(\gamma^1)^2}_{\substack{g_{11}\\=-1}}\underbrace{(\gamma^2)^2}_{\substack{g_{22}\\=-1}}\underbrace{(\gamma^3)^2}_{\substack{g_{33}\\=-1}} = \mathbb{1}$$

2. $$\mathrm{Tr}[\text{nb. impair de } \gamma^\mu] \overset{?}{=} 0$$
$$\mathrm{Tr}[\gamma^{\mu_1}\ldots\gamma^{\mu_n}] = \mathrm{Tr}[\gamma^{\mu_1}\ldots\gamma^{\mu_n}\overbrace{\gamma_5\gamma_5}^{=\mathbb{1}}] = \mathrm{Tr}[\overbrace{\gamma_5\gamma^{\mu_1}}^{=-\gamma^{\mu_1}\gamma_5 \text{ car } \{\gamma_5,\gamma^\mu\}=0}\ldots\gamma^{\mu_n}\gamma_5]$$
$$=(-1)^n\,\mathrm{Tr}[\gamma^{\mu_1}\ldots\gamma^{\mu_n}\gamma_5\gamma_5]$$
$$= 0 \quad \text{pour } n \text{ impair}$$

3. $$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = \frac{1}{2}\mathrm{Tr}[\underbrace{\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu}_{2g^{\mu\nu}\mathbb{1}}] = g^{\mu\nu}\,\mathrm{Tr}[\mathbb{1}] = 4g^{\mu\nu}$$

4. $$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu] \overset{!}{=} 0$$

Ansatz: $\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu] = c\,g^{\mu\nu}$ (la seule possibilité, coeff. c à determiner)

$$g_{\mu\nu}\cdot\Big|\quad \mathrm{Tr}[\gamma_5\underbrace{\gamma^\mu\gamma_\mu}_{=4}] = 4\,\mathrm{Tr}[\gamma_5] = 0 = c\,g^\mu_\mu = c\cdot 4 \Rightarrow c=0$$

$\llcorner$ $\{\gamma^\mu,\gamma_\mu\} = 2g^\mu_\mu$

$\underline{\mathrm{Tr}[\gamma_5 \cdot \text{moins que } 4\ \gamma^\mu] = 0}$ $\ulcorner$ $\mathrm{Tr}[\gamma_5 \cdot \text{nb impair de } \gamma^\mu] = 0$
car $\gamma_5 \hat{=}$ nb pair de γ^μ $\lrcorner$

5. $$\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = \mathrm{Tr}[(2g^{\mu\nu}-\gamma^\nu\gamma^\mu)\gamma^\rho\gamma^\sigma]$$
$$= 2g^{\mu\nu}\underbrace{\mathrm{Tr}[\gamma^\rho\gamma^\sigma]}_{=4g^{\rho\sigma}} - \mathrm{Tr}[\gamma^\nu(2g^{\mu\rho}-\gamma^\rho\gamma^\mu)\gamma^\sigma]$$
$$= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + \mathrm{Tr}[\gamma^\nu\gamma^\rho(2g^{\mu\sigma}-\gamma^\sigma\gamma^\mu)]$$
$$= 8g^{\mu\nu}g^{\rho\sigma} - 8g^{\mu\rho}g^{\nu\sigma} + 8g^{\mu\sigma}g^{\nu\rho} - \underbrace{\mathrm{Tr}[\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^\mu]}_{=\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma]}$$
$$\Rightarrow \underline{\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = 4(g^{\mu\nu}g^{\rho\sigma}+g^{\mu\sigma}g^{\nu\rho}-g^{\mu\rho}g^{\nu\sigma})}$$

6. Identité : $\gamma^\nu \gamma^\rho \gamma^\sigma \equiv S^{\nu\rho\sigma\alpha}\gamma_\alpha + i\epsilon^{\nu\rho\sigma\alpha}\gamma_\alpha\gamma_5$

$$\Rightarrow \mathrm{Tr}[\gamma_5 \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] = S^{\nu\rho\sigma\alpha}\underbrace{\mathrm{Tr}[\gamma_5\gamma^\mu\gamma_\alpha]}_{\overset{4.}{=}\,0} + i\epsilon^{\nu\rho\sigma\alpha}\,\mathrm{Tr}[\gamma_5\gamma^\mu\gamma_\alpha\gamma_5]$$

$$= i\epsilon^{\nu\rho\sigma\alpha}\underbrace{\mathrm{Tr}[\gamma^\mu\gamma_\alpha]}_{=4g^\mu{}_\alpha} = 4i\epsilon^{\nu\rho\sigma\mu} = \underline{4i\epsilon^{\mu\nu\rho\sigma}}$$

7. $\gamma_\mu\gamma_\nu\gamma^\mu = (2g_{\mu\nu} - \gamma_\nu\gamma_\mu)\gamma^\mu = 2\gamma_\nu - 4\gamma_\nu = \underline{-2\gamma_\nu}$

8. $\gamma_\mu\gamma_\nu\gamma_\sigma\gamma^\mu = (2g_{\mu\nu} - \gamma_\nu\gamma_\mu)\gamma_\sigma\gamma^\mu = 2\gamma_\sigma\gamma_\nu - \gamma_\nu\underbrace{(\gamma_\mu\gamma_\sigma\gamma^\mu)}_{\overset{7.}{=}-2\gamma_\sigma}$

$$= 2\gamma_\sigma\gamma_\nu + 2\gamma_\nu\gamma_\sigma = 2\{\gamma_\sigma,\gamma_\nu\} = \underline{4g_{\sigma\nu}}$$

9. $\gamma_\mu\gamma_\nu\gamma_\rho\gamma_\sigma\gamma^\mu = (2g_{\mu\nu} - \gamma_\nu\gamma_\mu)\gamma_\rho\gamma_\sigma\gamma^\mu = 2\gamma_\rho\gamma_\sigma\gamma_\nu - \gamma_\nu\underbrace{(\gamma_\mu\gamma_\rho\gamma_\sigma\gamma^\mu)}_{\overset{8.}{=}4g_{\rho\sigma}}$

$$= 2\gamma_\rho\gamma_\sigma\gamma_\nu - 4g_{\rho\sigma}\gamma_\nu$$

$$= 2(2g_{\rho\sigma} - \gamma_\sigma\gamma_\rho)\gamma_\nu - 4g_{\rho\sigma}\gamma_\nu$$

$$= \underline{-2\gamma_\sigma\gamma_\rho\gamma_\nu}$$

Ad 6. Ansatz le plus général :

$$\mathrm{Tr}[\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma] = a\,g^{\mu\nu}g^{\rho\sigma} + b\,g^{\mu\rho}g^{\nu\sigma} + c\,g^{\mu\sigma}g^{\nu\rho} + d\,\epsilon^{\mu\nu\rho\sigma}$$

Multiplication avec $g_{\mu\nu}g_{\rho\sigma}$, etc. $\Rightarrow a=b=c=0$

avec $\epsilon_{\mu\nu\rho\sigma}$ $\Rightarrow d = 4i$

(5.5) $\varepsilon^{\mu\alpha\beta\gamma}\,\varepsilon_{\mu}{}^{\rho\sigma\tau}$

$$= A\, g^{\alpha\rho} g^{\beta\sigma} g^{\gamma\tau} + B\, g^{\alpha\rho} g^{\beta\tau} g^{\gamma\sigma}$$
$$+ C\, g^{\alpha\sigma} g^{\beta\rho} g^{\gamma\tau} + D\, g^{\alpha\sigma} g^{\beta\tau} g^{\gamma\rho}$$
$$+ E\, g^{\alpha\tau} g^{\beta\rho} g^{\gamma\sigma} + F\, g^{\alpha\tau} g^{\beta\sigma} g^{\gamma\rho}$$

Contraction avec

Kontr. mit i) $g_{\alpha\rho}\, g_{\beta\sigma}\, g_{\gamma\tau}$ $\qquad (\varepsilon^{\mu\alpha\beta\gamma}\varepsilon_{\mu\alpha\beta\gamma} \equiv \varkappa,$

ii) $g_{\alpha\rho}\, g_{\beta\tau}\, g_{\gamma\sigma}$

iii) $g_{\alpha\sigma}\, g_{\beta\rho}\, g_{\gamma\tau}$

iv) $g_{\alpha\sigma}\, g_{\beta\tau}\, g_{\gamma\rho}$

v) $g_{\alpha\tau}\, g_{\beta\rho}\, g_{\gamma\sigma}$

vi) $g_{\alpha\tau}\, g_{\beta\sigma}\, g_{\gamma\rho}$:

i) $\varkappa = 64A + 16B + 16C + 4D + 4E + 16F$

ii) $-\varkappa = 16A + 64B + 4C + 16D + 16E + 4F$

iii) $-\varkappa = 16A + 4B + 64C + 16D + 16E + 4F$

iv) $\varkappa = 4A + 16B + 16C + 64D + 4E + 16F$

v) $\varkappa = 4A + 16B + 16C + 4D + 64E + 16F$

vi) $-\varkappa = 16A + 4B + 4C + 16D + 16E + 64F$

$\sum$ aller Gln / Σ de toutes les équations :

$$\Leftrightarrow \quad 0 = A + B + C + D + E + F \qquad (*)$$

i) + ii) , iii) + iv) , v) + vi) :

$$\Leftrightarrow \begin{cases} 0 = 4(A+B) + \underbrace{C + D + E + F}_{= -A-B \ (\text{s.} (*))} \\ 0 = A + B + 4(C+D) + E + F \\ 0 = \underbrace{A + B + C + D}_{= -E-F} + 4(E+F) \end{cases}$$

Alors :
Also :

$$B = -A$$
$$D = -C$$
$$F = -E$$

(löst alle bisherigen Gln.)

en i), iii), v) :

$$\Rightarrow \begin{cases} \mathcal{H} = 48A + 12C - 12E \\ -\mathcal{H} = 12A + 48C + 12E \\ \mathcal{H} = -12A + 12C + 48E \end{cases}$$

$$\Rightarrow \quad C = -A$$
$$E = A$$
$$A = \mathcal{H}/24$$

~~Also~~ Donc :

$$\boxed{\begin{aligned}\varepsilon^{\mu\alpha\beta\gamma}\,\varepsilon_{\mu}{}^{\rho\sigma\tau} = \frac{\mathcal{K}}{24}\Big(& g^{\alpha\rho}g^{\beta\sigma}g^{\gamma\tau} - g^{\alpha\rho}g^{\beta\tau}g^{\gamma\sigma} \\ & - g^{\alpha\sigma}g^{\beta\rho}g^{\gamma\tau} + g^{\alpha\sigma}g^{\beta\tau}g^{\gamma\rho} \\ & + g^{\alpha\tau}g^{\beta\rho}g^{\gamma\sigma} - g^{\alpha\tau}g^{\beta\sigma}g^{\gamma\rho}\Big)\end{aligned}}$$

Determination de ~~(Bestimmung von)~~ $\mathcal{K} = \varepsilon^{\mu\alpha\beta\gamma}\,\varepsilon_{\mu\alpha\beta\gamma}$:

$\varepsilon^{\mu\alpha\beta\gamma}$ (hat → a) $4! = 24$ nicht-verschwindende → éléments non-nuls Elemente (1 oder -1).

~~Ferner ist~~ De plus $\varepsilon^{\alpha\beta\gamma\delta} = g^{\alpha\lambda}g^{\beta\kappa}g^{\gamma\mu}g^{\delta\nu}\,\varepsilon_{\lambda\kappa\mu\nu}$

$= -\varepsilon_{\alpha\beta\gamma\delta}$ elementweise → Par élément

$\Rightarrow \varepsilon^{\mu\alpha\beta\gamma}\,\varepsilon_{\mu\alpha\beta\gamma} = -\sum$ (alle Elemente → tous les éléments)^2

$= -24$

$$\boxed{\mathcal{K} = -24}$$

$\Rightarrow$ ~~Es folgt~~:

$$\varepsilon^{\mu\alpha\beta\gamma}\,\varepsilon_{\mu\alpha}{}^{\sigma\tau} = -\Big((4-1-1)g^{\beta\sigma}g^{\gamma\tau} - (4-1-1)g^{\beta\tau}g^{\gamma\sigma}\Big)$$

$$= -2\left(g^{\beta\sigma}g^{\gamma\tau} - g^{\beta\tau}g^{\gamma\sigma}\right)$$

$$\varepsilon^{\mu\alpha\beta\gamma}\,\varepsilon_{\mu\alpha\beta}{}^{\tau} = -6\,g^{\gamma\tau}$$

$$\varepsilon^{\mu\alpha\beta\gamma}\,\varepsilon_{\mu\alpha\beta\gamma} = -24 \quad ✓$$