Physique des Particules I & II: Les groupes SU(n)

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Literature

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2) H. F. Jones, *Groups, Representations and Physics*, Taylor & Francis, New York

3) Hamermesh, *Group Theory*

4) Wu-Ki Tung, *Group Theory in Physics*, World Scientific

5) H. Georgi, *Lie algebras in particle physics*, Frontiers in Physics

6) Notes of V. Derya (Webpage I. Schienbein; Internships)

7) Robert Cahn, *Semi-Simple Lie Algebras and Their Representations*, freely available on internet

8) R. Slansky, *Group Theory for Unified Model Building*, Phys. Rep. 79 (1981) 1-128

One page summary of the world

Gauge group

$$\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$$

Particle content

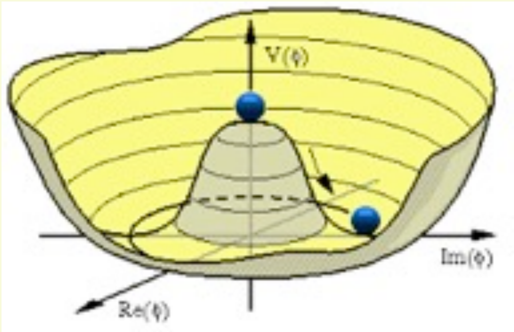
MATTER				HIGGS		GAUGE	
$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3},\mathbf{2})_{1/3}$	$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_{-1}$	$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_1$	B	$(\mathbf{1},\mathbf{1})_0$
u_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{-4/3}$	e_R^c	$(\mathbf{1},\mathbf{1})_2$			W	$(\mathbf{1},\mathbf{3})_0$
d_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{2/3}$	ν_R^c	$(\mathbf{1},\mathbf{1})_0$			G	$(\mathbf{8},\mathbf{1})_0$

Lagrangian
(Lorentz + gauge + renormalizable)

$$\mathcal{L} = -\frac{1}{4} G^{\alpha}_{\mu\nu} G^{\alpha\mu\nu} + \ldots \overline{Q}_k \not{D} Q_k + \ldots (D_\mu H)^\dagger (D^\mu H) - \mu^2 H^\dagger H - \frac{\lambda}{4!}(H^\dagger H)^2 + \ldots Y_{k\ell} \overline{Q}_k H (u_R)_\ell$$

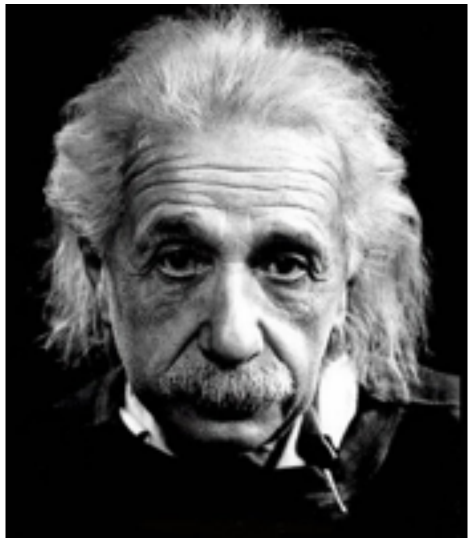
SSB

- $H \to H' + \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$
- $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \to \mathrm{U}(1)_Q$
- $B, W^3 \to \gamma, Z^0$ and $W^1_\mu, W^2_\mu \to W^+, W^-$
- Fermions acquire mass through Yukawa couplings to Higgs

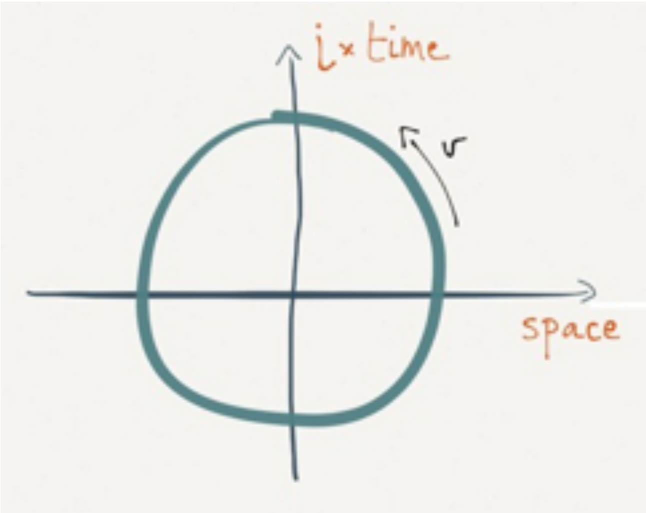


The general theoretical framework

Special relativity (SR)



- All inertial observers see the same physics:
  - same light speed c
  - Lorentz symmetries = Space-time "rotations"



$$x^\mu = (t, \vec{x})$$
$$x^2 = \eta_{\mu\nu} x^\mu x^\nu = x^\mu x_\mu = \text{invariant}$$
$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

- Energy-momentum relation: p = (E,**p**), $p^2 = m^2 = E^2 - \mathbf{p}^2$

Special relativity (SR)

- Lorentz group $O(1,3) = \{\Lambda \mid \Lambda^T \eta \Lambda = \eta\}$
- Proper Lorentz group $SO(1,3) = \{\Lambda \mid \Lambda^T \eta \Lambda = \eta, \det \Lambda = 1\}$
- Proper orthochronous Lorentz group $SO_+(1,3)$: $\Lambda_{00} \geq 1$
  Called the Lorentz group in the following
- Poincaré group = Inhomogeneous Lorenz group = $ISO_+(1,3)$

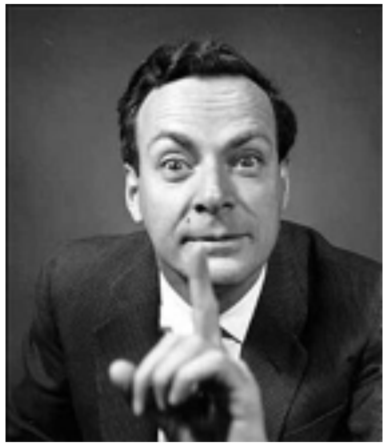
  $SO_+(1,3)$ and space-time Translations

Quantum Mechanics (QM)

- Determinism is not fundamental: $\Delta x^{\mu} \times \Delta p_{\nu} \geq (\hbar/2)\delta^{\mu}_{\nu}$
  - Nature is random → probability rules
  - The vacuum is not void, it fluctuates!

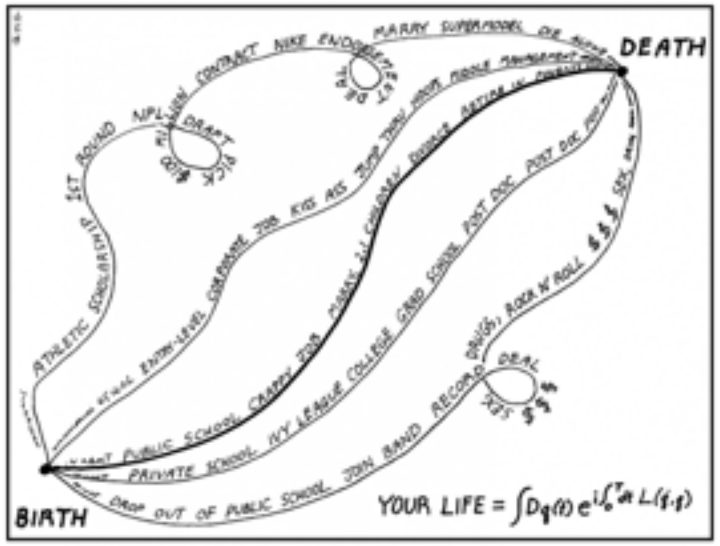
- Classical physics emerges from constructive interference of probability amplitudes:

Feynman's path integral:



$$A = \int [dq] \exp(iS[q(t), \dot{q}(t)])$$

a rational for the least action principle



The Path Integral Formulation of Your Life

Quantum Field Theory (QFT)

- The general theoretical framework in particle physics is Quantum Field Theory
- Weinberg I:

  QFT is the only way to reconcile quantum mechanics with special relativity

  "QFT = QM + SR"

Quantum Field Theory (QFT)

- **QM**: It's the same quantum mechanics as we know it!
- **SR**:
  - Relativistic wave equations are not sufficient!
    We need to change **number** and **types** of particles in particle reactions
  - Need **fields** and **quantize** them ("quantum fields")

Particles = Excitations (quanta) of fields

Symmetries I
(Lie groups, Lie algebras)

Symmetries are described by Groups

A group $(G, \odot)$ is a set of elements G together with an operation $\odot : G \times G \to G$ which satifies the following axioms:

- Associativity: $\forall a, b, c \in G : (a \odot b) \odot c = a \odot (b \odot c)$
- Neutral element: $\exists e \in G : \forall a \in G : e \odot a = a \odot e = a$
- Inverse element: $\forall a \in G : \exists a^{-1} \in G : a^{-1} \odot a = a \odot a^{-1} = e$

The group is called <u>commutative</u> or <u>Abelian</u> if also the following axiom is satisfied:

- Commutativity: $\forall a.b \in G : a \odot b = b \odot a$

Lie groups (simplified)

A Lie group is a group with the property that it depends differentiably on the parameters that define it.

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- Choose the parametrization such that $g(\vec{0}) = e$.

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Example:
Rotation $R(\phi) \in \mathrm{SO}(3)$ by an angle ϕ around the z-axis:

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Generators of a Lie group

Be $D(\vec{\alpha})$ an element of a n-dimensional Lie-group G, $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n)$.

We can do a Taylor expansion around $\vec{\alpha} = \vec{0}$ with $D(\vec{0}) = e$:

$$
\begin{aligned}
D(\vec{\alpha}) &= D(\vec{0}) + \sum_a \frac{\partial}{\partial \alpha_a} D(\vec{\alpha})_{|\vec{\alpha}=0} \alpha_a + \ldots \\
&= e + i \sum_a \alpha_a T^a + \ldots
\end{aligned}
$$

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The T^a $(a = 1, \ldots, n)$ are the generators of the Lie group:

$$T^a := -i \left[\frac{\partial}{\partial \alpha_a} D(\vec{\alpha}) \right]_{|\vec{\alpha}=0}$$

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The group element for general $\vec{\alpha}$ can be recovered
by exponentiation:

$$D(\vec{\alpha}) = \lim_{k \to \infty} (e + \sum_a \frac{i \alpha_a T^a}{k})^k = e^{i \sum_a \alpha_a T^a}$$

Lie algebra

- The generators T^a form a **basis** of a **Lie algebra**

Def.: A **Lie algebra g** is a vector space together with a skew-symmetric bilinear map [,]: g x g → g (called the Lie bracket) which satisfies the Jacobi identity

Lie algebra

- The generators T^a form a **basis** of a **Lie algebra**
- $[T^a, T^b] = i f^{ab}{}_c T^c$ (Einstein convention)
- The $f^{ab}{}_c$ are called **structure constants**

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Lie algebra

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- $[T^a, T^b] = i\, f^{ab}{}_c\, T^c$ (Einstein convention)
- The $f^{ab}{}_c$ are called **structure constants**
- Any group element connected to the neutral element can be generated using the generators:

  $\mathbf{g = \exp(i\, c_a\, T^a)}$ (Einstein convention)

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Rank

- Rank = Number of simultanesouly diagonalizable generators
- Rank = Number of good quantum numbers
- Rank = Dimension of the Cartan subalgebra

Symmetries II
(Representations)

Representations of a group

- Def.: A <u>linear representation</u> of a group G on a vector space V is a group homomorphism $D: G \to GL(V)$.

- Remarks:

  - $g \mapsto D(g)$, where $D(g)$ is a linear operator acting on V

  - The operators $D(g)$ preserve the group structure:
    $D(g_1\, g_2) = D(g_1)\, D(g_2)$, $D(e)$ = identity operator

  - V is called the <u>base space</u>, $\dim V$ = dimension of the representation

Representations of a group

- A representation (D,V) is reducible if a non-trivial subspace $U \subset V$ exists which is invariant with respect to D:

  $\forall g \in G: \forall \mathbf{u} \in U: D(g)\mathbf{u} \in U$

- A representation (D,V) is irreducible if it is not reducible

- A representation (D,V) is completely reducible if all $D(g)$ can be written in block diagonal form (with suitable base choice)

Representations of a Lie algebra

- Def.: A <u>linear representation</u> of a Lie algebra A on a vector space V is an algebra homomorphism $D: A \to \mathrm{End}(V)$.

- Remarks:

  - $t \mapsto T = D(t)$, where T is a linear operator acting on V

  - The operators $D(t)$ preserve the algebra structure:
    $[t^a, t^b] = i f^{ab}{}_c\, t^c \to [T^a, T^b] = i f^{ab}{}_c\, T^c$

  - A representation for the Lie algebra induces a representation for the Lie group

Tensor product

Composite systems are described mathematically by the **tensor product of representations**

- Tensor products of irreps are in general reducible!
- They are a direct sum of irreps: Clebsch-Gordan decomposition
- Examples:
  - System of two spin-1/2 electrons
  - Mesons: quark-anti-quark systems, Baryons: systems of three quarks

Symmetries III
(Space-time symmetries)

Space-time symmetry

- The minimal symmetry of a (relativistic) QFT is the **Poincaré symmetry**
- **Observables** should not change under Poincaré transformations of
    - Space-time coordinates $x = (t,\mathbf{x})$
    - Fields $\phi(x)$
    - States of the Hilbert space $|\mathbf{p}, \ldots\rangle$
- Need to know how the group elements are **represented** as operators acting on these objects (space-time, fields, states)
- At the classical level **Poincaré invariant Lagrangians** is all we need

Poincaré algebra I

- Poincaré group = Lorentz group $SO_+(1,3)$ + Translations
- Lorentz group has 6 generators: $J_{\mu\nu} = - J_{\nu\mu}$

  Lorentz algebra: $[J_{\mu\nu} , J_{\rho\sigma}]= -i (\eta_{\mu\rho} J_{\nu\sigma} - \eta_{\mu\sigma} J_{\nu\rho} - [\mu \leftrightarrow \nu])$
- Poincaré group has 10=6+4 generators: $J_{\mu\nu} , P_\mu$

  Poincaré algebra:
  $[P_\mu,P_\nu]=0,[J_{\mu\nu},P_\lambda]=i(\eta_{\nu\lambda} P_\mu - \eta_{\mu\lambda} P_\nu)$,Lorentz algebra

Poincaré algebra II

- Poincaré group has 10=6+4 generators: $J_{\mu\nu}$, P_μ
  - 3 Rotations → angular momentum $J_i=1/2\ \varepsilon_{ijk} J_{jk}$
    $[J_i,J_j] = i\ \varepsilon_{ijk}\ J_k$
  - 3 Boosts → $K_i = J_{0i}$
    $[K_i,K_j] = -i\ \varepsilon_{ijk}\ J_k$; $[J_i,K_j] = i\ \varepsilon_{ijk}\ K_k$
  - 4 Translations → energy/momentum P_μ
    $[J_i,P_j] = i\ \varepsilon_{ijk}\ P_k$, $[K_i,P_j] = -i\ \delta_{ij}\ P_0$, $[P_0,J_i] = 0$, $[P_0,K_i] = i\ P_i$

Tensor representations of so(1,3) (integer spin)

- All physical quantities can be classified according to their transformation properties under the Lorentz group

- Representations characterized by two invariants: **mass**, **spin** (Casimir operators P^2, W^2)

- Physical particles are irreps of the Poincaré group:

ϕ = scalar, V_μ = vector, $T_{\mu\nu}$ = tensor, ...

s=0 s=1 s=2

Spinor representations of so(1,3) (half integer spin)

- so(1,3) ~ sl(2,**C**) ~ $su(2)_L \times su(2)_R$

  $J_m^+ := J_m + i K_m$, $J_m^- := J_m - i K_m$: $\mathbf{[J_m^+ , J_n^-] = 0}$, $\mathbf{[J_i^+,J_j^+] = i\, \varepsilon_{ijk}\, J_k^+}$, $\mathbf{[J_i^-,J_j^-] = i\, \varepsilon_{ijk}\, J_k^-}$

- $su(2)_{L,R}$ labelled by $j_{L,R}$ = 0, 1/2, 1, 3/2, 2, ...
  - (j_L, j_R) = (0,0) scalar
  - (1/2,0) left-handed Weyl spinor; (0,1/2) right-handed Weyl spinor
  - (1/2,1/2) vector
- Dirac spinor = (1/2,0) + (0,1/2) is reducible (not fundamental)
  Note: (1/2,0) and (0,1/2) can have different interactions
- Majorana spinor = (1/2,0) + $(0,1/2)^c$ for neutral fermions only

Representation of so(1,3) on fields

- A field $\phi(x)$ is a function of the coordinates
- Lorentz transformation: $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu \, x^\nu$, $\phi \to \phi'$
- Scalar field: $\phi'(x') = \phi(x)$

  At the same time $\phi'(x') = \exp(i \; 1/2 \; \omega_{\mu\nu} \, J^{\mu\nu}) \; \phi(x)$

  Comparison allows to find a concrete expression for $J^{\mu\nu}$:
  $J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$ with $S^{\mu\nu}=0$, $L^{\mu\nu} = x^\mu P^\nu - x^\nu P^\mu$ where $P^\mu = i \, \partial^\mu$
- Similar procedure for Weyl, Dirac, Vector fields, ...
  and for the full Poincaré group

Symmetries IV
(Unitary symmetries)

Internal symmetries

- Coleman-Mandula theorem:

  The most general symmetry of a relativistic QFT:

  Space-time symmetry x Internal symmetry (**direct product**)

- Algebra: **direct sum** space-time generators and internal symmetry generators
  - 3 rotations
  - 3 boosts
  - 4 translations
  - generators T^a of internal symmetry

SU(n)

- Group: $SU(n) = \{U \in M_n(\mathbf{C}) \mid U^\dagger U = \mathbf{I}_n, \det U = 1\}$
- Algebra: $su(n) = \{t \in M_n(\mathbf{C}) \mid tr(t) = 0, t^\dagger = t\}$
- $\dim SU(n) = \dim su(n) = n^2-1$
- $\text{rank } su(n) = n-1$
- Important representations (D,V):
  - The fundamental representation: **n** (V is an n-dimensional vector space)
  - The anti-fundamental representation: **n***
  - The adjoint representation: $V = su(n)$, dimension of adjoint representation = n^2-1

SU(2)

- dim SU(2) = dim su(2) = $2^2-1 = 3$
- rank su(2) = 2-1 = 1
- Algebra: $[t_k,t_l]=i\,\varepsilon_{klm}\,t_m$
- The fundamental representation: **2**
  $T_i = 1/2\,\sigma_i$ (i=1,2,3), σ_i Pauli matrices
- irreps: Basis states $|j,j_z\rangle$, j=0,1/2,1,3/2,2,... ; j_z=-j, -j+1, ..., j-1,j

SU(3)

- dim SU(3) = dim su(3) = 3^2-1 = 8
- rank su(3) = 3-1 = 2
- Algebra: $[t_a,t_b]=i\, f_{abc}\, t_c$
- The fundamental representation: **3**
  $T_i = 1/2\ \lambda_i$ (i=1,2,3), λ_i Gell-Mann matrices
- The structure constants can be calculated using the generators in the fundamental irrep: $f_{abc} = -2i\, Tr([Ta,Tb]Tc)$
- irreps: labeled by 2 integer numbers (rank = 2)

Glossary of Group Theory: I. Basics

- Group
  - discrete, continuous, Abelian, non-Abelian
  - subgroup = subset which is a group
  - invariant subgroup = normal subgroup
  - simple group = has no *proper* invariant subgroups
- Lie group: continuous group which depends differentiably on its parameters
  - dimension = number of essential parameters
- Lie algebra
  - generators = basis of the Lie algebra; elements of the tangent space T_eG
  - dimension = number of linearly independent generators
  - structure constants = specifiy the algebra (basis dependent)
  - subalgebra = subset which is an algebra
  - ideal = invariant subalgebra
  - simple algebra = has no *proper* ideals (smallest building block; irreducible)
  - semi-simple algebra = direct sum of simple algebras

Glossary of Group Theory: II. Representations

- Representations
  - of groups
  - of algebras
  - equivalent, unitary, reducible, entirely reducible
  - irreducible representations (irreps)
  - fundamental representation
  - adjoint representation
- Direct sum of two representations
- Tensor product of two representations
  - Clebsch-Gordan decomposition
  - Clebsch-Gordan coefficients
- Quadratic Casimir operator
- Dynkin index

Glossary of Group Theory: III. Cartan-Weyl

- Cartan-Weyl analysis of simple Lie algebras: G = H ⊕ E
  - H = Cartan subalgebra = maximal Abelian subalgebra of G
  - rank G = dimension of Cartan subalgebra = number of simultaneously diagonalisable operators
  - E = space of ladder operators
  - Root vector (labels the ladder operators)
    - positive roots = if first non-zero component positive (basis dependent)
    - simple roots = positive root which is *not* a linear combination of other positive roots with positive coefficients
  - Weight vector (quantum numbers of the physical states)
    - heighest weight

Glossary of Group Theory: IV. Dynkin

- Dynkin diagrams
  - complete classification of all simple Lie algebras by Dynkin
  - Dynkin diagrams ↔ simple roots → roots → ladder operators
  - Dynkin diagrams ↔ simple roots → roots → geometrical interpretation of commutation relations
- Cartan matrix
  - Simple Lie algebra ↔ root system ↔ simple roots ↔ Dynkin diagrams ↔ Cartan matrix
- Dynkin lables (of a weight vector)
- Dynkin diagrams + Dynkin labels ⇒ recover whole algebra structure
  - analysis of any irrep of any simple Lie algebra (non-trivial in other notations)
  - tensor products
  - subgroup structure, branching rules

Groups and the Standard Model of particle physics

The general procedure

- **Introduce Fields & Symmetries**

The general procedure

- Introduce Fields & Symmetries
- **Construct a local Lagrangian density**

The general procedure

- Introduce Fields & Symmetries
- Construct a local Lagrangian density
- **Describe Observables**
  - How to measure them?
  - How to calculate them?

The general procedure

- Introduce Fields & Symmetries
- Construct a local Lagrangian density
- Describe Observables
  - How to measure them?
  - How to calculate them?
- **Falsify: Compare theory with data**

Fields & Symmetries

Matter content of the Standard Model (including the antiparticles)

Matter				Higgs		Gauge	
$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3}, \mathbf{2})_{1/3}$	$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$(\mathbf{1}, \mathbf{2})_{-1}$	$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$	$(\mathbf{1}, \mathbf{2})_1$	A	$(\mathbf{1}, \mathbf{1})_0$
u_R^c	$(\overline{\mathbf{3}}, \mathbf{1})_{-4/3}$	e_R^c	$(\mathbf{1}, \mathbf{1})_2$			W	$(\mathbf{1}, \mathbf{3})_0$
d_R^c	$(\overline{\mathbf{3}}, \mathbf{1})_{2/3}$	ν_R^c	$(\mathbf{1}, \mathbf{1})_0$			G	$(\mathbf{8}, \mathbf{1})_0$

$Q^c = \begin{pmatrix} u_L^c \\ d_L^c \end{pmatrix}$	$(\overline{\mathbf{3}}, \overline{\mathbf{2}})_{-1/3}$	$L^c = \begin{pmatrix} \nu_L^c \\ e_L^c \end{pmatrix}$	$(\mathbf{1}, \overline{\mathbf{2}})_1$	$H = \begin{pmatrix} h^- \\ h^0 \end{pmatrix}$	$(\mathbf{1}, \overline{\mathbf{2}})_{-1}$	A	$(\mathbf{1}, \mathbf{1})_0$
u_R	$(\mathbf{3}, \mathbf{1})_{4/3}$	e_R	$(\mathbf{1}, \mathbf{1})_{-2}$			W	$(\mathbf{1}, \mathbf{3})_0$
d_R	$(\mathbf{3}, \mathbf{1})_{-2/3}$	ν_R	$(\mathbf{1}, \mathbf{1})_0$			G	$(\mathbf{8}, \mathbf{1})_0$

Matter content of the Standard Model

- Left-handed up quark u_L:
  - LH Weyl fermion: $u_{L\alpha} \sim (1/2,0)$ of $so(1,3)$
  - a color triplet: $u_{Li} \sim 3$ of $SU(3)_c$
  - Indices: $(u_L)_{i\alpha}$ with $i=1,2,3$ and $\alpha=1,2$
- Similarly, left-handed down quark d_L
- u_L and d_L components of a $SU(2)_L$ doublet: $Q_\beta = (u_L\ ,\ d_L) \sim 2$
  - Q carries a hypercharge 1/3: $Q \sim (3,2)_{1/3}$ of $SU(3)_c \times SU(2)_L \times U(1)_Y$
  - Indices: $Q_{\beta i\alpha}$ with $\beta=1,2$; $i=1,2,3$ and $\alpha=1,2$

Matter content of the Standard Model

- There are three generations: $\mathbf{Q_k}$, k =1,2,3
- Lot’s of indices: $Q_{k\beta i\alpha}(x)$
- We know how the indices **β,i,α** transform under symmetry operations (i.e., which representations we have to use for the generators)

Matter content of the Standard Model

- Right-handed up quark u_R:
  - RH Weyl fermion: $u_{R\dot{\alpha}} \sim (0,1/2)$ of so(1,3)
  - a color triplet: $u_{Ri} \sim 3$ of $SU(3)_c$
  - a singlet of $SU(2)_L$: $u_R \sim 1$ (no index needed)
  - u_R carries hypercharge 4/3: $u_R \sim (3,1)_{4/3}$
  - Indices: $(u_R)_{i\dot{\alpha}}$ with i=1,2,3 and $\dot{\alpha}$=1,2 (Note the dot)
  - Note that $u_R^c \sim (3^*,1)_{-4/3}$

Matter content of the Standard Model

- Again there are three generations: u_{Rk} , k =1,2,3
- Lot's of indices: $u_{Rki\alpha.}(x)$
- And so on for the other fields ...

Terms for the Lagrangian

How to build Lorentz scalars?
Scalar field (like the Higgs)

Real field ϕ

$$\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$$

Complex field $\phi = \frac{1}{\sqrt{2}}(\varphi_1 + i\varphi_2)$

$$\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$$

Note: The mass dimension of each term in the Lagrangian has to be 4!

How to build Lorentz scalars?
Fermions (spin 1/2)

Left-handed Weyl spinor

$$i\psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L$$

Right-handed Weyl spinor

$$i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R$$

Mass term mixes left and right

$$i\psi_L^\dagger \overline{\sigma}^\mu \partial_\mu \psi_L + i\psi_R^\dagger \sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L)$$

$$\sigma^\mu = (1, \sigma^i)$$

$$\bar{\sigma}^\mu = (1, -\sigma^i)$$

Dirac spinor in chiral basis

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \qquad i\overline{\Psi}\gamma^\mu \partial_\mu \Psi - m\overline{\Psi}\Psi \quad \text{with} \quad \overline{\Psi} = \Psi^\dagger \gamma^0 \quad \text{and} \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix}$$

How to build Lorentz scalars? Vector boson (spin 1)

U(1) gauge boson ("Photon")

$$-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \quad \text{where} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Mass term allowed by Lorentz invariance; forbidden by gauge invariance

In principle, there is a second invariant

$$-\frac{1}{4}F_{\mu\nu}\widetilde{F}^{\mu\nu} \quad \text{with} \quad \widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

$$F\tilde{F} \propto \vec{E}\cdot\vec{B}$$

Violates Parity, Time reversal, and CP symmetry; prop. to a total divergence → doesn't contribute in QED

BUT strong CP problem in QCD

Gauge symmetry

- Idea: Generate interactions from free Lagrangian by imposing **local (i.e. $\alpha = \alpha(x)$) symmetries**
- Does not fall from heavens; generalization of 'minimal coupling' in electrodynamics
- Final judge is experiment: It works!

Local gauge invariance for a complex scalar field

$$\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi \quad \text{is invariant under } \phi \to e^{i\alpha}\phi.$$

What if now α = α(x) depends on the space-time?

$$\begin{aligned}
&\partial_\mu(e^{i\alpha(x)}\phi)^*\partial^\mu(e^{i\alpha(x)}\phi) - m^2(e^{i\alpha(x)}\phi)^*(e^{i\alpha(x)}\phi) \\
&= [\partial_\mu e^{i\alpha(x)} \cdot \phi + e^{i\alpha(x)} \cdot \partial_\mu\phi]^*[\partial^\mu e^{i\alpha(x)} \cdot \phi + e^{i\alpha(x)} \cdot \partial^\mu\phi] - m^2\phi^*\phi \\
&= [ie^{i\alpha(x)}\partial_\mu\alpha(x) \cdot \phi + e^{i\alpha(x)} \cdot \partial_\mu\phi]^*[ie^{i\alpha(x)}\partial^\mu\alpha(x) \cdot \phi + e^{i\alpha(x)} \cdot \partial^\mu\phi] - m^2\phi^*\phi \\
&= [-ie^{-i\alpha(x)}\partial_\mu\alpha(x) \cdot \phi^* + e^{-i\alpha(x)} \cdot \partial_\mu\phi^*][ie^{i\alpha(x)}\partial^\mu\alpha(x) \cdot \phi + e^{i\alpha(x)} \cdot \partial^\mu\phi] - m^2\phi^*\phi \\
&= -ie^{-i\alpha(x)}\partial_\mu\alpha(x) \cdot \phi^* \cdot ie^{i\alpha(x)}\partial^\mu\alpha(x) \cdot \phi \\
&\quad - ie^{-i\alpha(x)}\partial_\mu\alpha(x) \cdot \phi^* \cdot e^{i\alpha(x)} \cdot \partial^\mu\phi \\
&\quad + e^{-i\alpha(x)} \cdot \partial_\mu\phi^* \cdot ie^{i\alpha(x)}\partial^\mu\alpha(x) \cdot \phi \\
&\quad + e^{-i\alpha(x)} \cdot \partial_\mu\phi^* \cdot e^{i\alpha(x)} \cdot \partial^\mu\phi \\
&\quad - m^2\phi^*\phi \\
&= \partial_\mu\phi \cdot \partial^\mu\phi - m^2\phi^*\phi + \text{non-zero terms}
\end{aligned}$$

Not invariant under U(1)!

Local gauge invariance for a complex scalar field

Can we find a derivative operator that commutes with the gauge transformation?

Define

$$D_\mu = \partial_\mu + iA_\mu,$$

where the *gauge field* A_μ transforms as

$$A_\mu \to A_\mu - \partial_\mu \alpha$$

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$$\begin{aligned} D_\mu\phi &\to (\partial_\mu + i[A_\mu - \partial_\mu\alpha(x)])[e^{i\alpha(x)}\phi] \\ &= \partial_\mu[e^{i\alpha(x)}\phi] + i[A_\mu - \partial_\mu\alpha(x)][e^{i\alpha(x)}\phi] \\ &= ie^{i\alpha(x)}\partial_\mu\alpha(x)\cdot\phi + e^{i\alpha(x)}\partial_\mu\phi + iA_\mu e^{i\alpha(x)}\phi - i\partial_\mu\alpha(x)e^{i\alpha(x)}\phi \\ &= e^{i\alpha(x)}\partial_\mu\phi + iA_\mu e^{i\alpha(x)}\phi \\ &= e^{i\alpha(x)}[\partial_\mu\phi + iA_\mu]\phi \\ &= e^{i\alpha(x)}D_\mu\phi \end{aligned}$$

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&= ie^{i\alpha(x)}\partial_\mu \alpha(x) \cdot \phi + e^{i\alpha(x)}\partial_\mu \phi + iA_\mu e^{i\alpha(x)}\phi - i\partial_\mu \alpha(x) e^{i\alpha(x)}\phi \\
&= e^{i\alpha(x)}\partial_\mu \phi + iA_\mu e^{i\alpha(x)}\phi \\
&= e^{i\alpha(x)}[\partial_\mu \phi + iA_\mu]\phi \\
&= e^{i\alpha(x)} D_\mu \phi
\end{aligned}$$

Nota bene:

- We call D_μ the *covariant derivative*, because it transforms just like ϕ itself:

$$\phi \to e^{i\alpha(x)}\phi \quad \text{and} \quad D_\mu \phi \to e^{i\alpha(x)} D_\mu \phi$$

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$$D_\mu \phi^* D^\mu \phi - m^2 \phi^* \phi \to e^{-i\alpha(x)} D_\mu \phi^* \cdot e^{i\alpha(x)} D^\mu \phi - m^2 e^{-i\alpha(x)} \phi^* \cdot e^{i\alpha(x)} \phi = D_\mu \phi^* D^\mu \phi - m^2$$

Expanding the Lagrangian

$D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi$ invariant under local U(1) transformations

$$D_\mu\phi^* D^\mu\phi - m^2\phi^*\phi = \partial_\mu\phi^*\partial^\mu\phi + iA^\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) + \phi^*\phi A_\mu A^\mu - m^2\phi^*\phi$$

- Demand symmetry $\rightarrow$ Generate interactions
- Generated mass for gauge boson (after ϕ acquires a vacuum expectation value)
- Explicit mass term forbidden by gauge symmetry (although otherwise allowed):

$$m^2 A_\mu A^\mu \rightarrow m^2(A_\mu - \partial_\mu\alpha)(A_\mu - \partial_\mu\alpha) \neq m^2 A_\mu A^\mu$$

- Simplest form of Higgs mechanism
- Vector-scalar-scalar interaction

Non-Abelian gauge symmetry

Abelian	Non-Abelian: component notation	Non-Abelian: vector notation
$U = e^{i\alpha(x)}$	$U = e^{i\alpha^a(x)T^a_R}$	$U = e^{i\alpha^a(x)T^a_R}$
$\phi \to U\phi$	$\mathbf{\Phi}^i \to U^i{}_k \mathbf{\Phi}^k$	$\mathbf{\Phi} \to U\mathbf{\Phi}$
A_μ	$A^a_\mu T^a_R$	$\boldsymbol{A}_\mu$
$A_\mu \to A_\mu - \partial_\mu \alpha$	$A^a_\mu T^a \to U A^a_\mu T^a U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$	$\boldsymbol{A}_\mu \to U\boldsymbol{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$
$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$	$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$	$\boldsymbol{F}_{\mu\nu} := \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu + ig[\boldsymbol{A}_\mu, \boldsymbol{A}_\nu]$
$F_{\mu\nu} \to F_{\mu\nu}$		$\boldsymbol{F}_{\mu\nu} \to U\boldsymbol{F}_{\mu\nu}U^\dagger$
$F_{\mu\nu}$ invariant	$F^a_{\mu\nu}F^{a\mu\nu}$ invariant	$\mathrm{Tr}(\boldsymbol{F}_{\mu\nu}\boldsymbol{F}^{\mu\nu})$ invariant

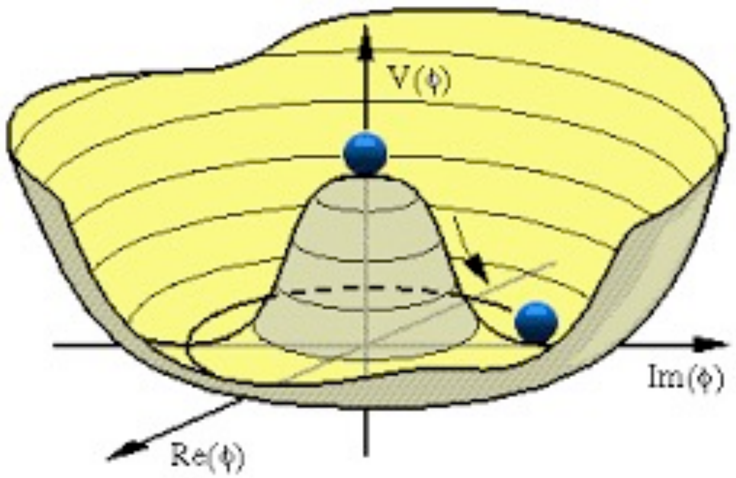
$$D_\mu = \partial_\mu + igA^a_\mu T^a_R$$

Conjecture

- All fundamental internal symmetries are gauge symmetries.
- Global symmetries are just "accidental" and not exact.

Spontaneous Symmetry Breaking

The Higgs mechanism



- The Higgs potential: $V = \mu^2\, \phi^\dagger\phi + \lambda\, (\phi^\dagger\phi)^2$
- Vacuum = Ground state = Minimum of V:
- If $\mu^2>0$ (massive particle): $\phi_{min} = 0$ (no symmetry breaking)
- If $\mu^2<0$: $\phi_{min} = \pm v = \pm(-\mu^2/\lambda)^{1/2}$
  These two minima in one dimension correspond to a continuum of minimum values in SU(2).
  The point $\phi = 0$ is now instable.
- Choosing the minimum (e.g. at +v) gives the vacuum a preferred direction in isospin space → spontaneous symmetry breaking
- Perform perturbation around the minimum

Higgs self-couplings

In the SM, the Higgs self-couplings are a consequence of the Higgs potential after expansion of the Higgs field $H \sim (1,2)_1$ around the vacuum expectation value which breaks the ew symmetry:

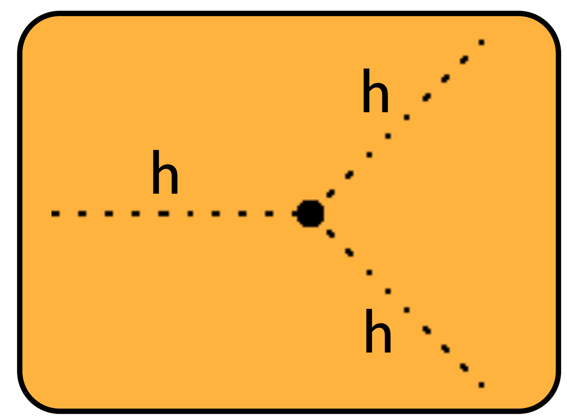
$$V_H = \mu^2 H^\dagger H + \eta (H^\dagger H)^2 \rightarrow \frac{1}{2} m_h^2 h^2 + \boxed{\sqrt{\frac{\eta}{2}} m_h h^3} + \boxed{\frac{\eta}{4} h^4}$$

with: $$m_h^2 = 2\eta v^2 \,,\; v^2 = -\mu^2/\eta$$

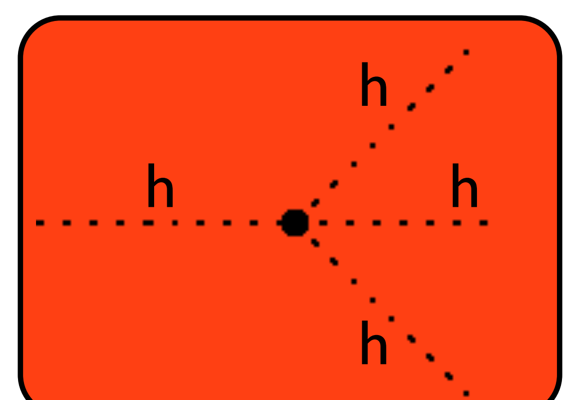
Note: v=246 GeV is fixed by the precision measures of G_F

In order to completely reconstruct the Higgs potential, on has to:

- Measure the 3h-vertex:
  via a measurement of Higgs pair production
  $$\lambda_{3h}^{\rm SM} = \sqrt{\frac{\eta}{2}} m_h$$



- Measure the 4h-vertex:
  more difficult, not accessible at the LHC in the high-lumi phase



One page summary of the world

Gauge group

$SU(3)_c \times SU(2)_L \times U(1)_Y$

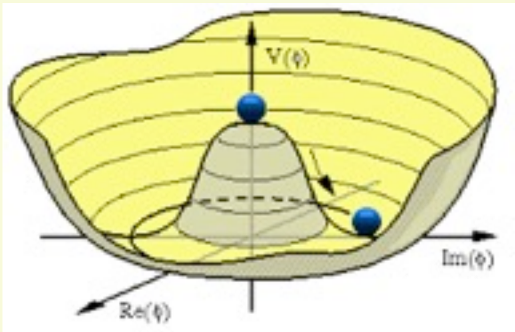
Particle content

Matter				Higgs		Gauge	
$Q = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$(\mathbf{3},\mathbf{2})_{1/3}$	$L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_{-1}$	$H = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$	$(\mathbf{1},\mathbf{2})_1$	B	$(\mathbf{1},\mathbf{1})_0$
u_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{-4/3}$	e_R^c	$(\mathbf{1},\mathbf{1})_2$			W	$(\mathbf{1},\mathbf{3})_0$
d_R^c	$(\overline{\mathbf{3}},\mathbf{1})_{2/3}$	ν_R^c	$(\mathbf{1},\mathbf{1})_0$			G	$(\mathbf{8},\mathbf{1})_0$

Lagrangian
(Lorentz + gauge + renormalizable)

$$\mathcal{L} = -\frac{1}{4}G^{\alpha}_{\mu\nu}G^{\alpha\mu\nu} + \ldots \overline{Q}_k \not{D} Q_k + \ldots (D_\mu H)^\dagger (D^\mu H) - \mu^2 H^\dagger H - \frac{\lambda}{4!}(H^\dagger H)^2 + \ldots Y_{k\ell}\overline{Q}_k H (u_R)_\ell$$

SSB



- $H \to H' + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix}$
- $SU(2)_L \times U(1)_Y \to U(1)_Q$
- $B, W^3 \to \gamma, Z^0$ and $W^1_\mu, W^2_\mu \to W^+, W^-$
- Fermions acquire mass through Yukawa couplings to Higgs

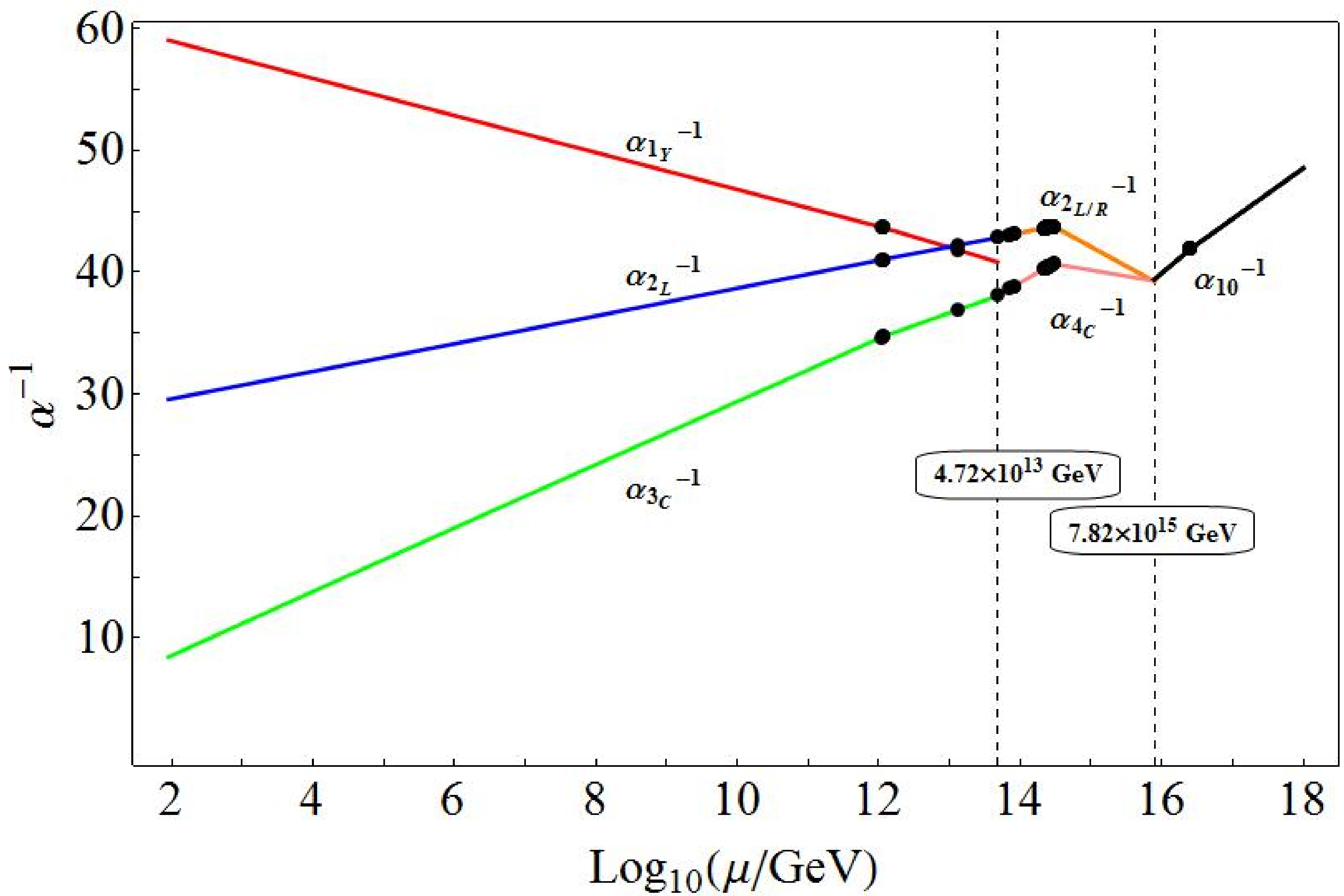
Grand Unified Theories

Aestethics, Symmetry, Religion

- Gauge symmetry SU(3) x SU(2) x U(1)
  - not a simple group
  - left-right asymmetric (maximal parity violation)
- Matter content in different representations
  - left vs right, quarks vs leptons
- Why three generations? (Why three space dimensions?)
  (“Who has ordered this?” Rabi after muon discovery)
- Wouldn’t it be a revelation to have complete **unification**?
  - one simple gauge group = one interaction
  - one representation for all matter = one matter type/one primary substance

Attractive features of GUTs

K. S. Babu, S. Khan, 1507.06712



- Gauge coupling unification
- Explanation for quantization of electric charges

(Some) GUT group candidates

- G_{SM} = SU(3) x SU(2) x U(1)
  - rank[G_{SM}] = rank[SU(3)] + rank[SU(2)] + rank[U(1)] = 2 +1 + 1 = 4
  - G_{SM} < G, where G is the gauge group of the GUT theory
  - rank[G_{SM}] $\leq$rank[G]
- Rank 4:
  - SU(5) unique rank 4 candidate: $\bar{5} + 10$
  - no ν_R, no B-L symmetry
- Rank 5:
  - SO(10): 16-plet
  - Pati-Salam group G(442) = $SU(4)_c$ x $SU(2)_L$ x SU(2)
- Rank 6:
  - E_6
  - Trinification $[SU(3)]^3$

Breaking patterns and branching rules

- Breaking patterns:
  - $SU(5) \to G_{SM} \to SU(3)_c \times U(1)_{em}$
  - $SO(10) \to SU(5) \to G_{SM} \to SU(3)_c \times U(1)_{em}$
  - $SO(10) \to G(442) \to G_{SM} \to SU(3)_c \times U(1)_{em}$
  - $E_6 \to SO(10) \to \ldots$
  - There are two aspects:
    - a) What are the subgroups of G with equal or lower rank?
    - b) Which Higgs fields are needed for the symmetry breaking?
- Branching rules:
  How does a multiplet of G split up into multiplets of G_{SM} after symmetry breaking?
  - Example $SU(5) \to G_{SM}$: $5 \to (3,1)_{2/5} + (1,2)_{-3/5}$