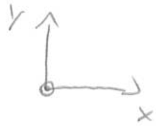
(6.1)

Repr. [2] : $R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$



$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ car $x \to x$, $y \to -y$

$$S\,R(\varphi)S^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} c\varphi & -s\varphi \\ s\varphi & c\varphi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} c\varphi & -s\varphi \\ -s\varphi & -c\varphi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} c\varphi & s\varphi \\ -s\varphi & c\varphi \end{pmatrix} = R(-\varphi)$$

$\Rightarrow$ $O(2)$ pas abélien (Abélien $\Rightarrow$ $SRS^{-1} = SS^{-1}R = R$)

Irreps de $SO(2)$: $D^{(m)}(\varphi) = e^{-im\varphi}$, $m \in \mathbb{Z}$ $\quad\left[\ D^{V} = D^{(1)} \oplus D^{(-1)}\ \right]$

$$S\,D^{(m)}(\varphi)\,S^{-1} = D^{(m)}(-\varphi) = e^{im\varphi} = D^{(-m)}(\varphi)$$ $\leadsto$ $D^{(m)}(\varphi)$ pas fermé pour $m \neq 0$

$\Rightarrow$ Irreps de $O(2)$: $D^{(m)}(\varphi) \oplus D^{(-m)}(\varphi)$ pour $m \neq 0$

$$D^{|m|}(\varphi) = D^{(m)}(\varphi) \oplus D^{(-m)}(\varphi) = \begin{pmatrix} e^{im\varphi} & 0 \\ 0 & e^{-im\varphi} \end{pmatrix}$$

$$D^{|m|}(S) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Caractères :

$$\chi^{|m|}(\varphi) = 2\cos m\varphi, \quad \chi^{|m|}(S) = 0$$

Pour $m = 0$: $D^{(0)}(\varphi) = 1$ reste irréductible

avec deux possibilités pour S : $S = 1$ ou $S = -1$

$\chi^{A_+}(\varphi) = 1$, $\chi^{A_+}(S) = 1$: repr. triviale

$\chi^{A_-}(\varphi) = 1$, $\chi^{A_-}(S) = -1$

$$D^{|m|}(\varphi) = \begin{pmatrix} e^{im\varphi} & 0 \\ 0 & e^{-im\varphi} \end{pmatrix}$$

$$\begin{pmatrix} x'+iy' \\ x'-iy' \end{pmatrix} = D^{|m|}(\varphi) \begin{pmatrix} x+iy \\ x-iy \end{pmatrix}$$

$$x'+iy' = e^{im\varphi}(x+iy)$$
$$x'-iy' = e^{-im\varphi}(x-iy)$$

$$2x' = (e^{im\varphi} + e^{-im\varphi})x + (e^{im\varphi} - e^{-im\varphi})iy$$
$$= 2\cos m\varphi\, x - 2\sin m\varphi\, y$$

$$2iy' = (e^{im\varphi} - e^{-im\varphi})x + (e^{im\varphi} + e^{-im\varphi})iy$$

$$x' = \cos m\varphi\, x - \sin m\varphi\, y$$
$$y' = \sin m\varphi\, x + \cos m\varphi\, y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = D'^{|m|}(\varphi)\begin{pmatrix} x \\ y \end{pmatrix}, \quad D'^{|m|}(\varphi) = \begin{pmatrix} \cos m\varphi & -\sin m\varphi \\ \sin m\varphi & \cos m\varphi \end{pmatrix}$$

$$\begin{pmatrix} x'+iy' \\ x'-iy' \end{pmatrix} = D^{|m|}(s)\begin{pmatrix} x+iy \\ x-iy \end{pmatrix}, \quad D^{|m|}(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$x'+iy' = x-iy$$
$$x'-iy' = x+iy$$

$$2x' = 2x \qquad \qquad x' = x$$
$$\Leftrightarrow$$
$$2iy' = -2iy \qquad \qquad y' = -y$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = D'^{|m|}(s)\begin{pmatrix} x \\ y \end{pmatrix}, \quad D'^{|m|}(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(6.2)

Angles d'Euler : $R(\varphi,\theta,\psi) = e^{-iX_3''\psi} \, e^{-iX_2'\theta} \, e^{-iX_3\varphi}$

$\uparrow$ axe en direction (θ,φ)

$S R_{\hat{n}}(\alpha) S^{-1} = R_{\hat{n}'}(\alpha)$ avec $\hat{n}' = S\hat{n}$

$X_3' = X_3$ car première rotation autour l'axe $\hat{e}_z'$ ne change pas l'axe $\hat{e}_z'$

$\Rightarrow \quad e^{iX_3''\psi} = e^{-iX_2'\theta} \, e^{-iX_3\psi} \, e^{iX_2'\theta}$

$\Rightarrow \quad R(\varphi,\theta,\psi) = e^{-iX_2'\theta} \, e^{-iX_3(\psi+\varphi)}$

Aussi : $\quad e^{-iX_2'\theta} = e^{-iX_3\varphi} \, e^{-iX_2\theta} \, e^{iX_3\varphi}$

$$\Rightarrow \boxed{R(\varphi,\theta,\psi) = e^{-iX_3\varphi} \, e^{-iX_2\theta} \, e^{-iX_3\psi}}$$

Forme standard pour la tabulation des matrices de rotation $D^{(j)}$:

$$D^{(j)}_{m'm}(\varphi,\theta,\psi) = e^{-im'\varphi} \, D^{j}_{m'm}(R_2(\theta)) \, e^{-im\psi}$$

Seule partie non-triviale :

$$d^{j}_{m'm}(\theta) = D^{j}_{m'm}(R_2(\theta)) \qquad \left(j = 0, 1/2, 1, 3/2, 2, \dots\right)$$

(6.3)

$X_2 = \frac{1}{2} G_2 , \quad G_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \quad G_2^2 = \mathbb{1}$

$$d^{1/2}(\theta) = e^{-iX_2\theta} = e^{-iG_2\frac{\theta}{2}} = \mathbb{1}\cos\frac{\theta}{2} - iG_2 \sin\frac{\theta}{2}$$

$\Big[\; e^{-iG_2 x} = 1 - iG_2 x - \frac{x^2}{2!} + iG_2 \frac{x^3}{3!} + \frac{x^4}{4!} \pm \dots$

$= \cos x - iG_2 \sin x \quad , x = \frac{\theta}{2} \; \Big]$

$$\Rightarrow d^{1/2}(\theta) = \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$$

(6.4)

Cours: $|jm\rangle = \sum_{m_1 m_2} \overbrace{\langle m_1 m_2 | jm\rangle}^{C^*(\ldots)} |m_1 m_2\rangle$

$$D^{(j)}_{m'm}(R) \equiv \langle jm' | U(R) | jm\rangle$$

$$= \sum_{m_1'} \sum_{m_1} C(j_1 j_2 j; m_1' \; m'-m_1' \; m') \, C^*(j_1 j_2 j; m_1 \; m-m_1 \; m) \, D^{(j_1)}_{m_1' m_1}(R) \, D^{(j_2)}_{m_2' m_2}(R)$$

$j_1 = j_2 = \frac{1}{2}$ $\qquad\qquad m_1, m_1' = -j_1, \ldots, j_1$

$j = 1$ $\qquad\qquad m_2, m_2' = -j_2, \ldots, j_2$

$R = R_z(\theta)$: $d^j_{m'm}(\theta) \equiv D^j_{m'm}(R_z(\theta))$; $m, m' = -j, \ldots, j$

$$d^1_{m'm}(\theta) = \sum_{m_1'} \sum_{m_1} C(\tfrac{1}{2} \tfrac{1}{2} 1; m_1' \; \overbrace{m'-m_1'}^{m_2'} \; m') \, C^*(\tfrac{1}{2} \tfrac{1}{2} 1; m_1 \; \overbrace{m-m_1}^{m_2} \; m) \, d^{1/2}_{m_1' m_1}(\theta) \, d^{1/2}_{m_2' m_2}(\theta)$$

$d^1_{11}, d^1_{10}, d^1_{1-1}$; $d^1_{01}, d^1_{00}, d^1_{0-1}$; $d^1_{-11}, d^1_{-10}, d^1_{-1-1}$

$$C(\tfrac{1}{2}\, \tfrac{1}{2}\, 1; \tfrac{1}{2}\, \tfrac{1}{2}\, 1) = C(\tfrac{1}{2}\, \tfrac{1}{2}\, 1; -\tfrac{1}{2}\, -\tfrac{1}{2}\, -1) = 1$$

$$C(\tfrac{1}{2}\, \tfrac{1}{2}\, 1; \tfrac{1}{2}\, -\tfrac{1}{2}\, 0) = C(\tfrac{1}{2}\, \tfrac{1}{2}\, 1; -\tfrac{1}{2}\, \tfrac{1}{2}\, 0) = \frac{1}{\sqrt{2}}$$

$$d^1_{11}(\theta) = \left[d^{1/2}_{1/2\, 1/2}(\theta) \right]^2 = \cos^2 \frac{\theta}{2} = \frac{1}{2}(1 + \cos\theta)$$

$$d^1_{10}(\theta) = d^{1/2}_{1/2\, 1/2}(\theta) \, d^{1/2}_{1/2\, -1/2}(\theta) \, \frac{1}{\sqrt{2}} \cdot 2 = \sqrt{2} \cos\frac{\theta}{2} \left(-\sin\frac{\theta}{2}\right) = -\frac{1}{\sqrt{2}} \sin\theta$$

$$d^1_{1-1}(\theta) = d^{1/2}_{1/2\, -1/2}(\theta) \, d^{1/2}_{1/2\, -1/2}(\theta) = \sin^2 \frac{\theta}{2} = \frac{1}{2}(1 - \cos\theta)$$

skip

$$d^1_{01}(\theta) = d^{1/2}_{1/2\, 1/2}(\theta) \, d^{1/2}_{-1/2\, 1/2}(\theta) \, \frac{1}{\sqrt{2}} \cdot 2 = -d^1_{10}(\theta) = -d^1_{0-1}(\theta)$$

$$d^1_{00}(\theta) = \frac{1}{2} \left(d^{1/2}_{1/2\, 1/2} \, d^{1/2}_{-1/2\, -1/2} + d^{1/2}_{1/2\, -1/2} \, d^{1/2}_{-1/2\, 1/2} \right) \cdot 2 = \cos^2 \frac{\theta}{2} - \sin^2 \theta/2 = \cos\theta$$

$$d^1_{-11}(\theta) = d^1_{1-1}(\theta)$$

$$d^1_{-10}(\theta) = \frac{1}{\sqrt{2}} \left(d^{1/2}_{-1/2\, 1/2} \, d^{1/2}_{-1/2\, -1/2} + \cancel{d^{1/2}_{-1/2\, -1/2}} \, \cancel{d^{1/2}_{-1/2\, 1/2}} \right) 2 = \sqrt{2} \sin\frac{\theta}{2} \cos\frac{\theta}{2} = \frac{1}{\sqrt{2}} \sin\theta = -d^1_{10}(\theta)$$

$$d^1_{-1-1} = d^1_{11}(\theta)$$

$$d^1_{m'm}(\theta) = \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & -\frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1-\cos\theta) \\ \frac{1}{\sqrt{2}}\sin\theta & \cos\theta & -\frac{1}{\sqrt{2}}\sin\theta \\ 1/2(1-\cos\theta) & \frac{1}{\sqrt{2}}\sin\theta & \frac{1}{2}(1+\cos\theta) \end{pmatrix}$$

$$D^V(R_2(\theta)) = \begin{pmatrix} \cos\theta & 0 & +\sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} = R_{ij}(\theta) \quad ; \quad x_i' = R_{ij}x_j$$

$$x' = c_\theta x + s_\theta z$$
$$y' = y$$
$$z' = -s_\theta x + c_\theta z$$

$$x'+iy' = c_\theta x + s_\theta z + iy = c_\theta (x+iy) + (x+iy) + s_\theta z - c_\theta iy - x$$

$$= (1+\cos\theta)(x+iy) + s_\theta z - c_\theta iy - x$$

$$= \frac{1}{2}(1+\cos\theta)(x+iy) + s_\theta z$$

$$+ \underbrace{\frac{1}{2}(1+\cos\theta)(x+iy) - x - \cos\theta\, iy}_{-\frac{1}{2}(1-\cos\theta)x + \frac{1}{2}(1-\cos\theta)iy}$$

$$= \frac{1}{2}(1+\cos\theta)(x+iy) + s_\theta z - \frac{1}{2}(1-\cos\theta)(x-iy)$$

$$-\frac{1}{\sqrt{2}}(x'+iy') = \frac{1}{2}(1+\cos\theta)\left[-\frac{x+iy}{\sqrt{2}}\right] - \frac{1}{\sqrt{2}}\sin\theta\, z + \frac{1}{2}(1-\cos\theta)\frac{x-iy}{\sqrt{2}}$$

Similaire pour les autres lignes.

6.5

$$\langle j m' | J_M | j m \rangle = C(1\ j\ j\ ; M\ m\ m') \langle j \| \vec{J} \| j \rangle$$

$$\langle j \| \vec{J} \| j \rangle \stackrel{!}{=} \sqrt{j(j+1)}$$

Prendre $m' = j$, $M = 0$ $\rightsquigarrow$ $J_0 = J_z$

$j \otimes 1$:

$$\underbrace{| j\, j \rangle}_{m'} = \alpha \underbrace{| j\, j \rangle}_{j_1 m_1} \underbrace{| 1\, 0 \rangle}_{j_2 m_2} + \beta\, | j\, j-1 \rangle | 1\, 1 \rangle$$

$$J_+ | j\, j \rangle = 0 = \alpha \sqrt{2}\, | j\, j \rangle | 1\, 1 \rangle + \beta \sqrt{2j}\, | j\, j \rangle | 1\, 1 \rangle$$

$J_+^{(2)} | 1\, 0 \rangle = \sqrt{(1-0)(1+1)}\, | 1\, 1 \rangle$

$J_+^{(1)} | j\, j-1 \rangle = \sqrt{(j-(j-1))(j+j)}\, | j\, j \rangle = \sqrt{2j}\, | j\, j \rangle$

$$\Rightarrow \alpha + \beta \sqrt{j} = 0$$

$$\Leftrightarrow \beta = -\frac{\alpha}{\sqrt{j}}$$

$J_+ | j\, m \rangle = \sqrt{(j-m)(j+m+1)}\, | j\, m+1 \rangle$

$$\langle j\, j | j\, j \rangle = 1 \Leftrightarrow \alpha^2 + \beta^2 = 1 = \alpha^2 + \frac{\alpha^2}{j} \Leftrightarrow \alpha = \frac{\sqrt{j}}{\sqrt{j+1}} \Rightarrow \beta = -\frac{1}{\sqrt{j+1}}$$

$$\Rightarrow | j\, j \rangle = \left(\sqrt{j}\, | j\, j \rangle | 1\, 0 \rangle - | j\, j-1 \rangle | 1\, 1 \rangle \right) \frac{1}{\sqrt{j+1}}$$

$$| j\, j \rangle = \sum_{\substack{m_1, m_2 \\ (m_1 + m_2 = j)}} \underbrace{\langle 1\, m_1, j\, m_2 | j\, j \rangle}_{C(1\ j\ j\ ; m_1\ m_2\ j)} | 1\, m_1 \rangle | j\, m_2 \rangle$$

$$\Rightarrow C(1\ j\ j\ ; 0\ j\ j) = \frac{\sqrt{j}}{\sqrt{j+1}}$$

$$C(1\ j\ j\ ; 1\ j-1\ j) = -\frac{1}{\sqrt{j+1}}$$

$M = 0$:

$$\langle j\, j | \underbrace{J_z | j\, j \rangle}_{j | j\, j \rangle} = j = \underbrace{C(1\ j\ j\ ; 0\ j\ j)}_{\frac{\sqrt{j}}{\sqrt{j+1}}} \langle j \| \vec{J} \| j \rangle$$

$$\Rightarrow \boxed{\langle j \| \vec{J} \| j \rangle = \sqrt{j(j+1)}}$$

Alternativement :

$M = 1$: $\quad J_1 = -(J_1 + i J_2)/\sqrt{2} = -\frac{1}{\sqrt{2}} J_+$

$$\langle j\, j \,|\, J_1 \,|\, j\, j-1 \rangle = \underbrace{C(1\; j\; j\,;\, 1\;\; j-1\;\; j)}_{= -\frac{1}{\sqrt{j+1}}} \langle j \,\|\, \vec{J} \,\|\, j \rangle$$

$$\| $$

$$-\frac{1}{\sqrt{2}} \sqrt{2j}\; \underbrace{\langle j j | j j \rangle}_{=1}$$

$J_+ |j\, j-1\rangle = \sqrt{2j}\; |j j\rangle$

$$-\sqrt{j} = -\frac{1}{\sqrt{j+1}} \langle j \,\|\, \vec{J} \,\|\, j \rangle$$

$$\Rightarrow \boxed{\langle j \,\|\, \vec{J} \,\|\, j \rangle = \sqrt{j(j+1)}} \quad \checkmark \quad \text{même résultat}$$