)

$D_3 = \{e, c, c^2, b, bc, bc^2\}$, $c^3 = (bc)^2 = e = b^2$

[2] rep. : $D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $D(c) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$, $D(c^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$

$D(b) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $D(bc) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$, $D(bc^2) = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$

↑
rotation autour
de l'axe x.

- $D^{\dagger} = D^{-1}$ $\quad [\, D(c)^T = D(c^2) \,]$
- TFO : $\sum_g D^{(\mu)}_{ir}(g)\, D^{(\nu)*}_{js}(g) = \frac{|G|}{n_\mu}\, \delta^{\mu\nu} \delta_{ij} \delta_{rs} =: \langle D^{(\mu)}_{ir}, D^{(\nu)}_{js} \rangle$
- Irreps of D_3 : A_1 , A_2 , $E \leftrightarrow [2]$-rep.
- 6-vectors $\{D_{ij}(g)\}$: $D(g) \equiv D^E(g)$

$\{D_{11}(g)\} = (1, -1/2, -1/2, 1, -1/2, -1/2)$

$\{D_{12}(g)\} = (0, -\sqrt{3}/2, \sqrt{3}/2, 0, -\sqrt{3}/2, \sqrt{3}/2)$

$\{D_{21}(g)\} = (0, \sqrt{3}/2, -\sqrt{3}/2, 0, -\sqrt{3}/2, \sqrt{3}/2)$

$\{D_{22}(g)\} = (1, -1/2, -1/2, -1, 1/2, 1/2)$

$\langle D^E_{ir}, D^E_{js} \rangle = \frac{6}{2} \delta_{ij}\delta_{rs} = 3\, \delta_{ij}\delta_{rs}$

Les irreps A_1, A_2 :

$\{D^{A_1}(g)\} = \{\chi^{A_1}(g)\} = (1, 1, 1, 1, 1, 1)$

$\{D^{A_2}(g)\} = \{\chi^{A_2}(g)\} = (1, 1, 1, -1, -1, -1)$

a)

$D(g_1)D(g_2) = D(g_1 g_2)$

$D^*(g) = (D(g))^*$

$D^*(g_1)D^*(g_2) = (D(g_1)D(g_2))^* = (D(g_1g_2))^* = D^*(g_1g_2) \Rightarrow D^*$ est une rep.

b)

Soit $D^*(g) = C^{-1}D(g)C \quad \forall g. \longrightarrow D^*(g)$

$\Rightarrow \quad D(g) = (C^*)^{-1} D^*(g)\, C^* = C^{*-1}\, \overbrace{C^{-1} D(g)\, C}^{D^*(g)}\, C^* = \underbrace{(CC^*)^{-1}}_{B^{-1}} D(g) \underbrace{CC^*}_{B} \quad \forall g$

$\Rightarrow$ Schur 1 : $B = CC^* = \lambda \mathbb{1} \quad , \lambda \in \mathbb{C}$

c) Soit D en plus unitaire : $D^+ = D^{-1} \quad \Rightarrow (D^*)^+ = (D^*)^{-1} \qquad$ car $(D^+)^* = (D^*)^+ ,\ (D^{-1})^* = (D^*)^{-1}$

b) $\overset{D^*=C^{-1}DC}{\leadsto} \quad (D^*)^{-1} = C^{-1}D^{-1}C \quad \checkmark |^+ \qquad (D^*)^{-1\,+} = D^*$

$\leadsto D^* = C^+ D (C^+)^{-1} \quad \checkmark \Rightarrow \quad D \overset{b)}{=} C D^* C^{-1} = CC^+ D \underbrace{C^{+-1}C^{-1}}_{(CC^+)^{-1}} = BDB^{-1}, B = CC^+$

$\Rightarrow$ Schur 1 : $CC^+ = \mu \mathbb{1} \qquad (CC^+)^+ = CC^+$ hermite $\Rightarrow \mu \in \mathbb{R}$

d)

$CC^+ = \mu \mathbb{1} \Rightarrow \underbrace{\det(CC^+)}_{=\mu^n} = (\det C)(\det C)^* = \underbrace{|\det C|^2}_{\neq 0 \text{ car } C \text{ inversible}} = \mu^n > 0 \Rightarrow \mu^n \in \mathbb{R}^+ \Rightarrow \mu \neq 0$

$\tilde{C} := \frac{1}{\sqrt{\mu}} C \Rightarrow \tilde{C}\tilde{C}^+ = \mathbb{1}$, c.à.d. $\tilde{C}$ unitaire

b) $\leadsto \tilde{C}\tilde{C}^* = \underset{\uparrow \lambda/\mu}{\tilde{\lambda}} \mathbb{1} \qquad \Rightarrow \tilde{C} = \tilde{\lambda}(\tilde{C}^*)^{-1} = \tilde{\lambda}\tilde{C}^T \quad \Rightarrow \tilde{C}^T = \tilde{\lambda}\tilde{C}$

$\Rightarrow \tilde{C} = \tilde{\lambda}^2 \tilde{C}$

$\Rightarrow \quad \tilde{\lambda} = \pm 1$

$\Rightarrow \quad \boxed{\tilde{C} = \pm \tilde{C}^T}$

4.3

$$D^{(\nu)}(h)\, B_i^\nu\, D^{(\nu)}(h)^{-1} = \sum_{g\in K_i} D^{(\nu)}(\underbrace{hgh^{-1}}_{g'}) = \sum_{g'\in K_i} D^{(\nu)}(g') = B_i^\nu \qquad \forall h\in G$$

$\Rightarrow$ B_i^ν commute avec tous les éléments d'une irrep: $[B_i^\nu, D^{(\nu)}(g)] = 0 \quad \forall g\in G$

$\Rightarrow$ Schur 1 : $B_i^\nu = \lambda_i^\nu \mathbb{1}$

(V G-espace vectoriel, $\dim V = n < \infty$)

$$\chi_i := \mathrm{Tr}\, B_i^\nu = \sum_{g\in K_i} \chi^\nu(g) = \kappa_i \chi_i^\nu = \lambda_i^\nu n_\nu \quad \Rightarrow \quad \chi_i^\nu = \frac{\lambda_i^\nu}{\kappa_i} n_\nu$$

Orthogonalité : $\langle \chi^{(\mu)}, \chi^{(\nu)} \rangle = \delta^{\mu\nu} = \frac{1}{|G|} \sum_i \kappa_i \chi_i^\nu \chi_i^{\mu *} \quad \Rightarrow \quad \sum_i \kappa_i |\chi_i^\nu|^2 = |G|$

$$\Rightarrow \quad \sum_i \kappa_i \left| \frac{\lambda_i^\nu}{\kappa_i} n_\nu \right|^2 = n_\nu^2 \sum_i \frac{|\lambda_i^\nu|^2}{\kappa_i} = |G|$$

$$\Rightarrow \quad \boxed{n_\nu^2 = \frac{|G|}{\sum_i |\lambda_i^\nu|^2 / \kappa_i}}$$

(4.4)

$$D = \sum_\nu a_\nu D^{(\nu)} \qquad a_\nu \in \mathbb{N}_0$$

$$\chi = \sum_\nu a_\nu \chi^{(\nu)} \qquad , \text{ pour les classes de conj.} \quad \chi_i = \sum_\nu a_\nu \chi_i^{(\nu)}$$

Frobenius : Groupe fini ✓

$$D \text{ irred.} \iff \langle \chi, \chi \rangle = 1$$

$$\langle \chi, \chi \rangle = \sum_{\nu,\mu} a_\mu a_\nu^* \underbrace{\langle \chi^{(\mu)}, \chi^{(\nu)} \rangle}_{\delta^{\mu\nu}} = \sum |a_\mu|^2$$

$\Rightarrow \quad \langle \chi, \chi \rangle = 1 \iff \exists! \mu : a_\mu = 1$ et $a_\nu = 0$ sinon

En détail :

$$|G| \langle \chi, \chi \rangle = \sum_i \kappa_i |\chi_i|^2 = \sum_i \kappa_i \sum_\mu a_\mu \chi_i^{(\mu)} \sum_\nu a_\nu^* \chi_i^{(\nu)*} = \sum_{\mu,\nu} a_\mu a_\nu^* \underbrace{\sum_i \kappa_i \chi_i^{(\mu)} \chi_i^{(\nu)*}}_{= |G| \delta^{\mu\nu}}$$

$$= |G| \sum_\mu |a_\mu|^2$$

D irred. $\iff$ un seul $a_\mu = 1$, les autres $a_\nu = 0$

$\rightsquigarrow \quad D$ irred $\iff \langle \chi, \chi \rangle = \sum_\mu |a_\mu|^2 = 1$

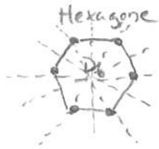
(4.5)

$N \trianglelefteq G$, rep. $D^{G/N} : G/N \to GL(V)$ peut être relevée :

$$D^G : G \to GL(V) \quad , \quad D^G(g) := D^{G/N}(gN)$$

$$\Rightarrow \quad \chi^G(g) = \chi^{G/N}(\underbrace{[g]}_{=gN})$$

$\underline{D_6}$: $C_2 \trianglelefteq D_6$, $D_6/C_2 \cong D_3$



$c : 60°$ [Lagrange : $|G| = |G:H|\,|H|$ ← classes à gauche

$12 = 6\cdot 2 = 4\cdot 3 = 3\cdot 4 = 2\cdot 6$]

$$D_6 = \{e, c, \dots, c^5, b, bc, \dots, bc^5\} \quad , \quad c^6 = b^2 = (bc)^2 = e$$

Classes de conj. : $[a] = \{ b \mid b = gag^{-1}, g \in G \}$

Partition : $K_1 = E = \{e\}$, $K_2 = 2C_6 = \{c, c^5\}$, $K_3 = 2C_6^2 = \{c^2, c^4\}$

$K_4 = C_6^3 = \{c^3\}$, $K_5 = 3C_2 = \{b, bc^2, bc^4\}$, $K_6 = 3C_2' = \{bc, bc^3, bc^5\}$

Exemple : $(bc)^2 = bcbc = e \Rightarrow cbc = b \Rightarrow \underline{cb = bc^5} \Rightarrow bcb = c^5$: $c^5 \sim c$

$c^2 b = c\,cb = c\,bc^5 = \underbrace{cbc}_{b}\,c^4 = bc^4 \Rightarrow bc^2 b = c^4$: $c^4 \sim c^2$

Co-ensembles : $|D_6| = |D_6 : C_2|\,|C_2|$: $12 = 6\cdot 2$ ⇝ 6 co-ensembles avec deux éléments

$gH = \{gh \mid h \in G\}$: $gC_2 = \{gh \mid h \in C_2\}$, $C_2 = \{e, c^3\}$, $[g] = \{g, gc^3\}$

D_6	D_6/C_2
K_1, K_4	E
K_2, K_3	C, C^2
K_5, K_6	B, BC, BC^2

$E := \{e, c^3\} = [e]$, $C := c^2 E = \{c^2, c^5\} = [c^2]$, $C^2 = c^4 E = \{c^4, c\} = [c^4]$

$B := bE = \{b, bc^3\} = [b]$, $BC = \{bc^2, bc^5\} = [bc^2]$, $BC^2 = \{bc, bc^4\} = [bc]$

Table de caractères de D_3 : (cours)

D_3	E ↔ E $K_1 = \{e\}$	$2C_3$ ↔ $2C_6$ $K_2 = \{c, c^2\}$	$3C_2''$ ↔ K_5 $K_3 = \{b, bc, bc^2\} = \{bc^3, bc^4, bc^5\} = \{b, bc^4, bc^2\}$	
A_1	1	1	1	
A_2	1	1	-1	z
E	2	-1	0	x, y

$$D_6/C_2 = \{E, C, C^2, B, BC, BC^2\}$$
$$\cong$$
$$D_3 = \{e, c, c^2, b, bc, bc^2\}$$

$D_6/C_2 \cong D_3$, $[g] \leftrightarrow \tilde{g}$

$$\chi_i^{D_6} \leftrightarrow \chi^{D_6}(g) = \chi^{D_6/C_2}([g]) = \chi^{D_3}(\tilde{g}) \leftrightarrow \chi_j^{D_3}$$

classes d'équiv. — co-ensemble — classes d'équiv.

D_6		D_3
A_1	$\leftrightarrow$	A_1
A_2	$\leftrightarrow$	A_2
E_2	$\leftrightarrow$	E

Pourquoi pas $E_1 \leftrightarrow E$?

D_6	E	$2C_6$	$2C_6^2$	C_6^3	$3C_2$	$3C_2'$
A_1	1	1	1	1	1	1
A_2	1	1	1	1	−1	−1
E_2	2	−1	−1	2	0	0
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
	E	$2C_3$	$2C_3$	E	$2C_3$	$2C_3$

D_3	$\{e\}$ = E	$\{c, c^2\}$ = $2C_3$	$\{b, bc, bc^2\}$ = $3C_2$
A_1	1	1	1
A_2	1	1	−1
E	2	−1	0

$$\chi^{D_6}_E = \chi^{D_6}(e) = \chi^{D_6/C_2}(E) = \chi^{D_3}(e) = \chi^{D_3}_E$$

$$\chi^{D_6}_{2C_6} = \chi^{D_6}(c) = \chi^{D_6/C_2}(C^2) = \chi^{D_3}(c^2) = \chi^{D_3}_{2C_3}$$
$$= \chi^{D_6}(c^5) = \chi^{D_6/C_2}(C) = \chi^{D_3}(c) = \chi^{D_3}_{2C_3}$$

$$\chi^{D_6}_{2C_6^2} = \chi^{D_6}(c^2) = \chi^{D_6/C_2}(C) = \chi^{D_3}_{2C_3}$$

$$\chi^{D_6}_{C_6^3} = \chi^{D_6}(c^3) = \chi^{D_6/C_2}(E) = \chi^{D_3}_E$$

$$\chi^{D_6}_{3C_2} = \chi^{D_6}(b) = \chi^{D_6/C_2}(B) = \chi^{D_3}(b) = \chi^{D_3}_{3C_2}$$

$$\chi^{D_6}_{3C_2'} = \chi^{D_3}_{3C_2}$$