EXAMEN 2014/2015

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1. **Permutations.**

   Soient

   $$p_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 1 & 4 & 6 & 7 & 2 & 3 & 5 & 10 & 8 \end{pmatrix}, \; p_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 3 & 8 & 4 & 1 & 6 & 5 & 2 & 7 & 9 \end{pmatrix}$$

   deux permutations.

   (a) Déterminez les produits $p_1 \cdot p_2$, $p_2 \cdot p_1$ et les symétriques p_1^{-1} et p_2^{-1}.

   (b) Écrivez p_1 et p_2 en notation cyclique.

   (c) Quel est l'ordre de p_1 et p_2 ?

2. **Le groupe D_3.**

   (a) Donnez la définition du groupe D_3. Quel est son ordre ? Interprétez les éléments du groupe comme transformations d'un triangle régulier non-orienté.

   (b) Spécifiez les classes de conjugaison.

   (c) Construisez la table de caractères. (Justifiez toutes les entrées dans la table.)

   (d) Calculez la décomposition Clebsch-Gordan pour la représentation $D^{(V)} \otimes D^{(V)}$ où $D^{(V)}$ est la représentation vectorielle (3-dimensionelle).

3. **Coefficients Clebsch-Gordan.**

   En utilisant les expressions pour les matrices $d^{1/2}_{m'm}(\theta)$ et $d^{1}_{m'm}(\theta)$ dérivez les fonctions $d^{3/2}_{3/2,3/2}(\theta)$, $d^{3/2}_{3/2,1/2}(\theta)$ et $d^{3/2}_{1/2,1/2}(\theta)$.

   Les identités trigonométriques suivantes peuvent être utiles :

   $$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}\,, \; \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,, \; \cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\,.$$

4. **Le groupe symplectique.**

   Le groupe symplectique est défini par $\mathrm{Sp}(2m, \mathbf{R}) := \{A \in \mathrm{GL}(2m, \mathbf{R}) | A^\dagger J A = J\}$ avec

   $$J := \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}\,.$$

   (a) Quelle est la dimension réelle du groupe général linéaire $\mathrm{GL}(2m, \mathbf{R})$, $\dim_{\mathbf{R}}\{\mathrm{GL}(2m, \mathbf{R})\}$?

   (b) Déterminez le nombre de contraintes à cause de la relation $A^\dagger J A = J$.
   Piste : Écrivez

   $$A = \begin{pmatrix} C & D \\ E & F \end{pmatrix}$$

   avec des matrices réelles $m \times m$ C, D, E, F.

   (c) Déterminez alors la dimension du groupe symplectique, $\dim_{\mathbf{R}}\{\mathrm{Sp}(2m, \mathbf{R})\}$.

Bon courage !

43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

	J	J ...
	M	M ...
m_1 m_2		
m_1 m_2 ⋮	Coefficients	

$$Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1 = -\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0 = \sqrt{\frac{5}{4\pi}}\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

$$Y_2^1 = -\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2 = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

$$Y_\ell^{-m} = (-1)^m Y_\ell^{m*}$$

$$d^\ell_{m,0} = \sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\, e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle = (-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^j_{m',m} = (-1)^{m-m'} d^j_{m,m'} = d^j_{-m,-m'}$$

1/2×1/2

m_1	m_2	J=1, M=+1
+1/2	+1/2	1

m_1	m_2	J=1, M=0	J=0, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=1, M=−1
−1/2	−1/2	1

1×1/2

m_1	m_2	J=3/2, M=+3/2
+1	+1/2	1

m_1	m_2	J=3/2, M=+1/2	J=1/2, M=+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	J=3/2, M=−1/2	J=1/2, M=−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	J=3/2, M=−3/2
−1	−1/2	1

2×1/2

m_1	m_2	J=5/2, M=+5/2
+2	+1/2	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	J=5/2, M=−5/2
−2	−1/2	1

3/2×1/2

m_1	m_2	J=2, M=+2
+3/2	+1/2	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	J=2, M=0	J=1, M=0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	J=2, M=−1	J=1, M=−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	J=2, M=−2
−3/2	−1/2	1

2×1

m_1	m_2	J=3, M=+3
+2	+1	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	J=3, M=−2	J=2, M=−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	J=3, M=−3
−2	−1	1

1×1

m_1	m_2	J=2, M=+2
+1	+1	1

m_1	m_2	J=2, M=+1	J=1, M=+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	J=2, M=0	J=1, M=0	J=0, M=0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	J=2, M=−1	J=1, M=−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	J=2, M=−2
−1	−1	1

3/2×1

m_1	m_2	J=5/2, M=+5/2
+3/2	+1	1

m_1	m_2	J=5/2, M=+3/2	J=3/2, M=+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	J=5/2, M=−3/2	J=3/2, M=−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	J=5/2, M=−5/2
−3/2	−1	1

3/2×3/2

m_1	m_2	J=3, M=+3
+3/2	+3/2	1

m_1	m_2	J=3, M=+2	J=2, M=+2
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	J=3, M=+1	J=2, M=+1	J=1, M=+1
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	J=3, M=−2	J=2, M=−2
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	J=3, M=−3
−3/2	−3/2	1

2×3/2

m_1	m_2	J=7/2, M=+7/2
+2	+3/2	1

m_1	m_2	J=7/2, M=+5/2	J=5/2, M=+5/2
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	J=7/2, M=+3/2	J=5/2, M=+3/2	J=3/2, M=+3/2
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	J=7/2, M=+1/2	J=5/2, M=+1/2	J=3/2, M=+1/2	J=1/2, M=+1/2
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	J=7/2, M=−1/2	J=5/2, M=−1/2	J=3/2, M=−1/2	J=1/2, M=−1/2
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	J=7/2, M=−3/2	J=5/2, M=−3/2	J=3/2, M=−3/2
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	J=7/2, M=−5/2	J=5/2, M=−5/2
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	J=7/2, M=−7/2
−2	−3/2	1

2×2

m_1	m_2	J=4, M=+4
+2	+2	1

m_1	m_2	J=4, M=+3	J=3, M=+3
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	J=4, M=+2	J=3, M=+2	J=2, M=+2
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	J=4, M=+1	J=3, M=+1	J=2, M=+1	J=1, M=+1
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=0	J=3, M=0	J=2, M=0	J=1, M=0	J=0, M=0
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	J=4, M=−1	J=3, M=−1	J=2, M=−1	J=1, M=−1
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	J=4, M=−2	J=3, M=−2	J=2, M=−2
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	J=4, M=−3	J=3, M=−3
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	J=4, M=−4
−2	−2	1

d functions

$$d^1_{0,0} = \cos\theta \qquad d^{1/2}_{1/2,1/2} = \cos\frac{\theta}{2} \qquad d^{1/2}_{1/2,-1/2} = -\sin\frac{\theta}{2}$$

$$d^1_{1,1} = \frac{1+\cos\theta}{2} \qquad d^1_{1,0} = -\frac{\sin\theta}{\sqrt{2}} \qquad d^1_{1,-1} = \frac{1-\cos\theta}{2}$$

$$d^{3/2}_{3/2,3/2} = \frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,1/2} = -\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-1/2} = \sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{3/2,-3/2} = -\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{3/2}_{1/2,1/2} = \frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{3/2}_{1/2,-1/2} = -\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^2_{2,2} = \left(\frac{1+\cos\theta}{2}\right)^2$$

$$d^2_{2,1} = -\frac{1+\cos\theta}{2}\sin\theta$$

$$d^2_{2,0} = \frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^2_{2,-1} = -\frac{1-\cos\theta}{2}\sin\theta$$

$$d^2_{2,-2} = \left(\frac{1-\cos\theta}{2}\right)^2$$

$$d^2_{1,1} = \frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^2_{1,0} = -\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^2_{1,-1} = \frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^2_{0,0} = \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

Figure 43.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

1.)

$$P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 1 & 4 & 6 & 7 & 2 & 3 & 5 & 10 & 8 \end{pmatrix}$$

$$P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 3 & 8 & 4 & 1 & 6 & 5 & 2 & 7 & 9 \end{pmatrix}$$

a)

$$P_1 \cdot P_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 8 & 4 & 5 & 6 & 9 & 2 & 7 & 1 & 3 & 10 \end{pmatrix} = (1\ 8)(2\ 4\ 6)(3\ 5\ 9)$$

$$P_2 \cdot P_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 10 & 4 & 6 & 5 & 3 & 8 & 1 & 9 & 2 \end{pmatrix} = (1\ 7\ 8)(2\ 10)(3\ 4\ 6)$$

$$P_1^{-1} = \begin{pmatrix} 9 & 1 & 4 & 6 & 7 & 2 & 3 & 5 & 10 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 6 & 7 & 3 & 8 & 4 & 5 & 10 & 1 & 9 \end{pmatrix} = (1\ 2\ 6\ 4\ 3\ 7\ 5\ 8\ 10\ 9)$$

$$P_2^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 8 & 2 & 4 & 7 & 6 & 9 & 3 & 10 & 1 \end{pmatrix} = (1\ 5\ 7\ 9\ 10)(2\ 8\ 3)$$

b)

$$P_1 = (1\ 9\ 10\ 8\ 5\ 7\ 3\ 4\ 6\ 2)$$

$$P_2 = (1\ 10\ 9\ 7\ 5)(2\ 3\ 8)(4)(6)$$

$$= (1\ 10\ 9\ 7\ 5)(2\ 3\ 8)$$

(c)

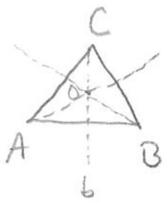
$$|P_1| = 10$$

$$|P_2| = \text{ppmc}(5,3) = 15$$

2.)

a) $D_3 = \langle c, b \rangle = \{e, c, c^2, b, bc, bc^2\}$ avec $c^3 = b^2 = (bc)^2 = e$

$|D_3| = 6$ $\Rightarrow cb = bc^2$



c : rotation par $120°$ dans le plan (anti-horaire)

$c = (ABC)$

b : rotation par π autour de l'axe $\overline{OC}$

$b = (AB)$

b) Classes de conjugaison :

$K_1 = \{e\} = E$, $K_2 = \{c, c^2\} = 2C_3$, $K_3 = \{b, bc, bc^2\} = 3C_2$

$bcb = b^2c^2 = c^2 \Rightarrow c \sim c^2$

c)

* $r = k$: nombre d'irreps = nombre de classes de conj. $\Rightarrow$ Il y a 3 irreps

* $|D_3| = 6 = n_1^2 + n_2^2 + n_3^2$; toujours $n_1 = 1$ (repr. triviale)

$\Rightarrow n_2^2 + n_3^2 = 5 \Rightarrow n_2 = 1, n_3 = 2$

$\Rightarrow$

D_3	E	$2C_3$	$3C_2$
A_1	1	1	1
A_2	1		
E	2		

Première ligne : $\chi_i = 1$ (repr. triv.)
Première colonne : dimensions

* irrep A_2 : $\dim A_2 = 1 \Rightarrow \chi(g_1 \cdot g_2) = \chi(g_1) \cdot \chi(g_2)$, $g_1, g_2 \in D_3$

$\chi(c) \underset{\substack{\uparrow \\ \text{même classe} \\ \text{de conj.}}}{=} \chi(c^2) = \chi(c)^2 \Rightarrow \boxed{\chi(c) = 1}$

$\chi(b^2) = \chi(e) = 1 = \chi(b)^2 \Rightarrow \chi(b) = \pm 1$

$\chi(b) = +1$ repr. triviale A_1
$\boxed{\chi(b) = -1}$ repr. A_2

$\Rightarrow$

D_3	E	$2C_3$	$3C_2$
A_1	1	1	1
A_2	1	1	-1
E	2	α	β

Orthogonalité : ligne 3, ligne 1 $\Rightarrow 0 = (1 \cdot 1^* \cdot 2 + 2 \cdot 1^* \cdot \alpha + 3 \cdot 1^* \cdot \beta)$

ligne 3, ligne 2 $\Rightarrow 0 = (1 \cdot 1^* \cdot 2 + 2 \cdot 1^* \cdot \alpha + 3 (-1)^* \cdot \beta)$

$$\Leftrightarrow \left.\begin{array}{l} 2\alpha + 3\beta + 2 = 0 \\ \wedge\ 2\alpha - 3\beta + 2 = 0 \end{array}\right\} \Leftrightarrow \left.\begin{array}{l} 4\alpha + 4 = 0 \\ \wedge\ 2\alpha + 3\beta + 2 = 0 \end{array}\right\} \Leftrightarrow \begin{array}{l} \alpha = -1 \\ \beta = 0 \end{array}$$

$\Rightarrow$

D_3	E	$2C_3$	$3C_2$
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

d) Repr. vectorielle : $\chi^V = \{3, 0, -1\}$

$$\underbrace{D^{(V)} \otimes D^{(V)}}_{=: D^{(V \times V)}} = a_1 A_1 \oplus a_2 A_2 \oplus a_3 E$$

$$\chi^{V \times V} = \chi^V \cdot \chi^V = \{9, 0, 1\}$$

$$a_1 = \langle \chi^{V \times V}, \chi^{A_1} \rangle = \frac{1}{6}\left[1 \cdot 9 \cdot 1 + 2 \cdot 0 \cdot 1 + 3 \cdot 1 \cdot 1\right] = 2$$

$$a_2 = \langle \chi^{V \times V}, \chi^{A_2} \rangle = \frac{1}{6}\left[1 \cdot 9 \cdot 1 + 0 + 3 \cdot 1 \cdot (-1)\right] = 1$$

$$a_3 = \langle \chi^{V \times V}, \chi^{E} \rangle = \frac{1}{6}\left[1 \cdot 9 \cdot 2 + 0 + 3 \cdot 1 \cdot 0\right] = 3$$

$$\Rightarrow \boxed{D^{(V)} \otimes D^{(V)} = 2 A_1 \oplus A_2 \oplus 3 E}$$

Check: $3 \cdot 3 = 2 + 1 + 3 \cdot 2 = 9$ ✓

3.)

j_1, j_2 fixe

$$d^{J}_{m'm}(\theta) = \langle J\, m' \,|\, R_2(\theta) \,|\, J\, m \rangle$$

$$= \sum_{\substack{m_1', m_2' \\ m_1, m_2}} \underbrace{\langle J\, m' \,|\, m_1'\, m_2' \rangle}_{\text{(coeff. C.-G.)}^*} \underbrace{\langle m_1'\, m_2' \,|\, R_2(\theta) \,|\, m_1\, m_2 \rangle}_{d^{j_1}_{m_1' m_1}(\theta)\, d^{j_2}_{m_2' m_2}(\theta)} \underbrace{\langle m_1 m_2 \,|\, J\, m \rangle}_{\text{Coeff. C.-G.}}$$

$m_1' + m_2' = m'$

$m_1 + m_2 = m$

$$\boxed{j_1 = 1,\ j_2 = 1/2}$$

$$d^{3/2}_{3/2, 3/2}(\theta) = \underbrace{\langle 3/2\ 3/2 \,|\, 1\ 1/2 \rangle}_{=1} \underbrace{\langle 1\ 1/2 \,|\, 3/2\ 3/2)}_{=1} d^{1}_{11}(\theta)\, d^{1/2}_{1/2\, 1/2}(\theta)$$

J, m', m

$$= \frac{1+\cos\theta}{2} \cdot \cos\frac{\theta}{2} \qquad \checkmark$$

$$d^{3/2}_{3/2, 1/2}(\theta) = \underbrace{\langle 3/2\ 3/2 \,|\, 1\ 1/2 \rangle}_{=1} \underbrace{\langle 1\ {-1/2} \,|\, 3/2\ 1/2 \rangle}_{=\frac{1}{\sqrt{3}}} d^{1}_{11}\, d^{1/2}_{1/2\, -1/2}$$

$$+ \underbrace{\langle 3/2\ 3/2 \,|\, 1\ 1/2 \rangle}_{=1} \underbrace{\langle 0\ 1/2 \,|\, 3/2\ 1/2 \rangle}_{=\sqrt{2/3}} d^{1}_{10}\, d^{1/2}_{1/2\, 1/2}$$

$$= \frac{1}{\sqrt{3}} \frac{1+\cos\theta}{2} \left(-\sin\frac{\theta}{2}\right) + \sqrt{\frac{2}{3}} \frac{-\sin\theta}{\sqrt{2}} \cos\frac{\theta}{2}$$

$$= -\frac{1}{\sqrt{3}} \left[\frac{1+\cos\theta}{2} \sin\frac{\theta}{2} + \underbrace{\sin\theta}_{=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} \cos\frac{\theta}{2} \right]$$

$$= -\frac{1}{\sqrt{3}} \sin\frac{\theta}{2} \left[\frac{1+\cos\theta}{2} + \underbrace{2\cos^2\frac{\theta}{2}}_{\frac{1+\cos\theta}{2}} \right] = -\frac{3}{\sqrt{3}} \sin\frac{\theta}{2} \frac{1+\cos\theta}{2}$$

$$= -\sqrt{3} \sin\frac{\theta}{2} \frac{1+\cos\theta}{2}$$

$$d^{3/2}_{1/2,1/2}(\theta) = \langle 3/2\ 1/2 \,|\, \substack{1 \\ 0}\ \substack{-1/2 \\ 1/2} \rangle \langle \substack{1 \\ 0}\ \substack{-1/2 \\ 1/2} \,|\, 3/2\ 1/2 \rangle\, d^{1}_{m_1' m_1}\, d^{1/2}_{m_2' m_2}$$

$$= \underbrace{\langle 3/2\ 1/2 \,|\, 1\ -1/2 \rangle}_{1/\sqrt{3}} \langle 1\ -1/2 \,|\, 3/2\ 1/2 \rangle\, d^{1}_{11}\, \underbrace{d^{1/2}_{-1/2\,-1/2}}_{d^{1/2}_{1/2,1/2}}$$

$$+ \langle 3/2\ 1/2 \,|\, 1\ -1/2 \rangle \langle 0\ 1/2 \,|\, 3/2\ 1/2 \rangle\, d^{1}_{10}\, \underbrace{d^{1/2}_{-1/2\ 1/2}}_{-d^{1/2}_{1/2,-1/2}}$$

$$+ \langle 3/2\ 1/2 \,|\, 0\ 1/2 \rangle \langle 1\ -1/2 \,|\, 3/2\ 1/2 \rangle\, d^{1}_{01}\, d^{1/2}_{1/2\ -1/2}$$

$$+ \underbrace{\langle 3/2\ 1/2 \,|\, 0\ 1/2 \rangle}_{\sqrt{2/3}} \langle 0\ 1/2 \,|\, 3/2\ 1/2 \rangle\, d^{1}_{00}\, d^{1/2}_{1/2\ 1/2}$$

$$= \frac{1}{3}\ \frac{1+\cos\theta}{2}\ \cos\frac{\theta}{2}$$

$$d^1_{01} = -d^1_{10}$$

$$+ \frac{1}{\sqrt{3}}\ \frac{\sqrt{2}}{\sqrt{3}}\ \frac{-\sin\theta}{\sqrt{2}}\ \sin\frac{\theta}{2}$$

$$+ \sqrt{\frac{2}{3}}\ \frac{1}{\sqrt{3}}\ \frac{\sin\theta}{\sqrt{2}}\ \left(-\sin\frac{\theta}{2}\right)$$

$$+ \sqrt{\frac{2}{3}}\ \sqrt{\frac{2}{3}}\ \cos\theta\ \cos\frac{\theta}{2}$$

$$= \frac{1}{3}\ \frac{1+\cos\theta}{2}\ \cos\frac{\theta}{2} - \underbrace{\frac{1}{3}\sin\theta\ \sin\frac{\theta}{2}} - \frac{1}{3}\sin\theta\ \sin\frac{\theta}{2} + \frac{2}{3}\cos\theta\ \cos\frac{\theta}{2}$$

$$= \frac{1}{3}\cos\frac{\theta}{2}\left[\frac{1+\cos\theta}{2} + 2\cos\theta\right] - \frac{2}{3}\underbrace{\sin\frac{\theta}{2}\ \sin\theta}$$

$$2\cos\frac{\theta}{2}\sin^2\frac{\theta}{2} = 2\cos\frac{\theta}{2}\left(1-\cos^2\frac{\theta}{2}\right)$$

$$= 2\cos\frac{\theta}{2}\left(1 - \frac{1+\cos\theta}{2}\right)$$

$$= \frac{1}{3}\cos\frac{\theta}{2}\left[\underbrace{\frac{1+\cos\theta}{2} + 2\cos\theta - 4 + 4\,\frac{1+\cos\theta}{2}}\right]$$

$$\frac{1}{2}\left(1+\cos\theta + 4\cos\theta - 8 + 4 + 4\cos\theta\right)$$

$$= \frac{1}{2}\left(-3 + 9\cos\theta\right)$$

$$= \cos\frac{\theta}{2}\ \frac{3\cos\theta - 1}{2} \quad \checkmark$$

4.)

$$Sp(2m,\mathbb{R}) = \{ A \in GL(2m,\mathbb{R}) \mid A^+ J A = J \} \quad \text{avec} \quad J := \begin{array}{cc} \begin{array}{cc} m & m \end{array} & \\ \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix} & \begin{array}{l} \}\, m \\ \}\, m \end{array} \end{array}$$

a) $\dim_{\mathbb{R}} \{ GL(2m,\mathbb{R}) \} = (2m)^2 = 4m^2$

b) Contraintes :

$$A = \begin{pmatrix} C & D \\ E & F \end{pmatrix} , \quad A^+ = \begin{pmatrix} C^+ & E^+ \\ D^+ & F^+ \end{pmatrix}$$

[avec $C^+ = C^T$, $D^+ = D^T$, $E^+ = E^T$, $F^+ = F^T$ car matrices réelles]

$$A^+ J A = \begin{pmatrix} C^+ & E^+ \\ D^+ & F^+ \end{pmatrix} \begin{pmatrix} E & F \\ -C & -D \end{pmatrix}$$

$$= \begin{pmatrix} C^+E - E^+C & C^+F - E^+D \\ D^+E - F^+C & D^+F - F^+D \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$$

$\Rightarrow$ (i) $C^+E = E^+C = (C^+E)^+ \quad \Rightarrow \quad \sum_{i=1}^{m-1} i = \frac{(m-1)m}{2}$ contraintes

(ii) $\left. \begin{array}{l} C^+F - E^+D = I_m \\ F^+C - D^+E = I_m \end{array} \right\} \Rightarrow m^2$ contraintes

(iii) $D^+F = F^+D = (D^+F)^+ \quad \Rightarrow \quad \frac{(m-1)m}{2}$ contraintes

c)

$$\dim_{\mathbb{R}} \{ Sp(2m,\mathbb{R}) \} = 4m^2 - \frac{(m-1)\cdot m}{2} \cdot 2 - m^2$$

$$= 3m^2 - m^2 + m = 2m^2 + m$$

$$= m(2m+1)$$