From Two-body Resonances to Three-body Borromean States*

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Abstract. In this work we use the complex scaled hyperspheric adiabatic expansion method to investigate the spectrum of three-body systems. This method provides a description of not only bound states, but also three-body resonances. We first consider the case of an infinitely heavy core and two non-interacting light particles. When each of the core-particle subsystems shows a resonance, the three-body state presents a three-body resonance with energy equal to the sum of the two two-body resonance energies. When one of the core-particle subsystems has a virtual $s$-state the sum of the two two-body energies does not correspond to a three-body resonance, but to a virtual state of the same energy relative to the resonant two-body system. The effects produced by the motion of the two-body center of masses when the mass of the core is made finite are also investigated. Calculations including an interaction between particles 2 and 3 are also considered. We show how these effects can lead to the appearance of Borromean three-body states.

1 Introduction

Along the years a large effort has been put to understand the properties of three-body systems, both for short and long range interactions. Three-body properties for positive energies, and in particular for three-body resonances, are much less established than for bound states, and the question of how these structures arise from the basic two-body interactions is still open. The numerical calculation of continuum wave functions and resonances unavoidably presents the problem of an asymptotically diverging wave function. Computation of three-body resonances requires then an accurate procedure to describe the three-body system, but also an efficient method to obtain the energy at which the $S$-matrix has a pole.

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A suitable method to describe spatially extended and weakly bound three-body systems was introduced in [1]. In this method the Faddeev equations are solved in coordinate space by means of a hyperspherical adiabatic expansion. This method reproduces accurately the asymptotics of the wave functions as proved by the fact that it derives the Efimov effect in which an accurate computation of the wave functions at very large distances is essential. A detailed description of the method and applications to atomic and nuclear physics can be found in [2].

A very efficient method to calculate resonances, introduced in the early 70’s, is the complex scaling method [3], where the radial coordinates are rotated into the complex plane by an angle \( \theta \) \((r \to re^{i\theta})\). When \( \theta \) is larger than the argument of a resonance then the corresponding divergent wave function becomes convergent, and the rotated wave function shows up after solving the corresponding Schrödinger equation in the same way as a bound state. The details of this method and its application to atomic collisions can be found in [4, 5].

In [6, 7] the complex scaling method together with the hyperspheric adiabatic expansion method was used to investigate three-body resonances in a variety of systems like \(^{11}\text{Li}, ^{6}\text{He}\) and the three-alpha’s system. In all the cases the method proved to be very efficient. In this work we employ the same procedure to investigate three-body resonances for a few simple schematic cases, making then possible to connect these resonances with the internal two-body structures, and trace how the three-body resonances evolve in the energy plane when more ingredients are introduced into the three-body system.

2 Complex Scaled Hyperspheric Adiabatic Expansion Method

To describe a three–body system we write the three-body wave function as a sum of three components \( \psi^{(i)}(x_i, y_i) \), each of them written in terms of each of the three possible sets of Jacobi coordinates [2]. These three components satisfy the three Faddeev equations

\[
(T - E)\psi^{(i)}(x_i, y_i) + V_{jk}(x_i) \left( \psi^{(i)}(x_i, y_i) + \psi^{(j)}(x_j, y_j) + \psi^{(k)}(x_k, y_k) \right) = 0, \quad (1)
\]

where \( T \) is the kinetic energy operator, \( V_{jk}(x_i) \) is the two-body interaction between particles \( j \) and \( k \), and \( E \) is the three-body energy.

By rewriting the Faddeev equations (1) in terms of the hyperspheric coordinates \((\rho = \sqrt{x^2 + y^2}, \alpha_i = \arctan x_i/y_i, \Omega_{x_i}, \text{and} \Omega_{y_i})\) it is possible now to separate them into angular and radial parts:

\[
\hat{\Lambda}^2 \phi_n^{(i)} + \frac{2m\rho^2}{\hbar^2} V_{jk}^{(i)}(x_i) \left( \phi_n^{(i)} + \phi_n^{(j)} + \phi_n^{(k)} \right) = \lambda_n(\rho)\phi_n^{(i)} \quad \text{(2)}
\]

\[
\left[ -\frac{d^2}{d\rho^2} - \frac{2mE}{\hbar^2} + \frac{1}{\rho^2} \left( \lambda_n(\rho) + \frac{15}{4} \right) \right] f_n(\rho) + \sum_{n'} \left( -2P_{nn'} \frac{d}{d\rho} - Q_{nn'} \right) f_{n'}(\rho) = 0 \quad \text{(3)}
\]

where \( \hat{\Lambda} \) is an angular operator [2], \( m \) is the normalization mass, \( n \) labels the angular eigenfunctions \( \phi_n^{(i)} \) and the angular eigenvectors \( \lambda_n \), and the functions \( P_{nn'} \) and \( Q_{nn'} \) can be found for instance in [2].
Finally, each of the components of the wave function is expanded in terms of the angular eigenfunctions $\phi_n^{(i)}$

$$\psi^{(i)}(x_i, y_i) = \frac{1}{\rho^{n/2}} \sum_n f_n(\rho) \phi_n^{(i)}(\rho, \Omega_i)$$

(4)

where $\Omega_i \equiv \{\alpha_i, \Omega_{xi}, \Omega_{yi}\}$ denote the five hyperspheric angular coordinates.

The procedure is then to solve the eigenvalue problem (2) and obtain the radial coefficients in the expansion (4) by solving the coupled set of differential equations (3) where the eigenvalues $\lambda_n(\rho)$ enter as effective potentials. Usually the expansion (4) converges rather fast, and only a few terms (typically no more than three) are needed.

To apply the complex scaling method the Jacobi coordinates $x_i$ and $y_i$ have to be rotated into the complex plane by an arbitrary angle $\theta$ ($x_i \rightarrow x_i e^{i\theta}$, $y_i \rightarrow y_i e^{i\theta}$). This means that only the hyperspheric radius $\rho$ is rotated ($\rho \rightarrow \rho e^{i\theta}$), while the five hyperangles remain unchanged. It is known that after this rotation the radial wave function of a resonance behaves asymptotically like

$$f_{nn'} \rightarrow e^{-|\kappa|\rho \sin (\theta - \theta_R)} e^{i|\kappa|\rho \cos (\theta - \theta_R)-K\pi/2+3\pi/4},$$

(5)

and therefore, as soon as $\theta$ is larger than the argument of the resonance $\theta_R$ the radial wave function falls off exponentially, exactly as a bound state. Thus, after complex scaling, the same numerical techniques used to compute bound states can be used for resonances. In particular, after solving the complex scaled equations (2) and (3) with the boundary condition in Eq.(5) one obtains simultaneously three-body resonances and bound states. Continuum states are rotated by an angle $2\theta$ in the energy plane [4].

3 Two resonances

We first apply the method to a simple system made by an infinitely heavy core and two light particles (in principle non identical). We start by assuming that the two light particles do not interact each other, while the core interacts with each of them via central potentials. For this particular system the three-body hamiltonian can be written as the sum of the two independent two-body hamiltonians. Let us consider now that each of the two core-particle interactions has a resonance at the energies $E_{12}$ and $E_{13}$ for the two-body subsystems 12 and 13, respectively. The separability of the three-body hamiltonian makes that when the two light particles populate the two-body resonances the energy of the three-body system is given precisely by the sum of the two two-body energies ($E = E_{12} + E_{13}$).

Although a three-body energy $E$ does not imply that the two-body resonances are populated it is obvious that this precise value of the three-body energy can have some particularities. In fact, after applying the method described in the previous section we have observed that the three-body system has a resonance at precisely the energy $E$. The calculation has been performed assuming two two-body $p$-resonances at the energies indicated by the two crosses in Fig.1. The light particles have masses equal $3M$ and $M$, respectively, $M$ being the nucleon mass. The sum of the two two-body resonance energies is indicated in the figure by
the thick solid circle. The coupling of the two two-body relative orbital angular momenta equal to 1 leads to three three-body states \((0^+, 1^+, \text{ and } 2^+)\) all the three degenerate at the energy \(E\).

When the mass of particle 1 is made finite the two-body center of masses 12 and 13 do not coincide anymore with the three-body center of mass. The motion of the two-body center of masses modifies the three-body energy and breaks the degeneracy of the three states. This is shown in Fig.1 by the solid circles, solid squares, and solid triangles, that show how the \(0^+, 1^+, \text{ and } 2^+\) three-body resonances evolve when \(M_1\) is made smaller. All the three curves start at the solid thick circle corresponding to \(M_1=\infty\) and end up at the point where \(M_1=3M, M_2=1.5M, \text{ and } M_3=\infty\). All along the three curves the reduced masses of the two-body systems 12 and 13 is kept constant, meaning that the position of the two-body resonance energies (crosses in the figure) is not changing.

As an example we show now how the three-body state corresponding to one of the systems in Fig.1 is moving when an interaction between particles 2 and 3 is included in the calculation. We have chosen the \(0^+\) state when \(M_1 = 4M\). An attractive \(s\)-wave interaction between particles 2 and 3 is making the energy and the width of the corresponding \(0^+\) state smaller and smaller, making at some point the three-body system bound. The two-body interaction is not binding.
any of the two-body subsystems, and the bound three-body states correspond to Borromean states. In this particular case we can see how when $M_1$ changes from $\infty$ to $4M$ (and $V_{23}=0$) a small change in the position of the three-body resonance is produced, being the interaction between particles 2 and 3 the main responsible of the eventual three-body binding of the system with $M_1=4M$.

4 One virtual state and one resonance

Let us consider now the same three-body system as before ($M_1=\infty$, $M_2=M$, $M_3=3M$, and no interaction between particles 2 and 3), but where one of the two-body subsystems has a $p$-resonance at the energy $E_{13}$ and the other one a virtual $s$-state at $E_{12}$. Again these states are indicated by crosses in the figure (Fig.2), and the three-body energy $E=E_{12}+E_{13}$ is shown by the thick solid circle.

Figure 2. $1^-$ three-body resonances for a three-body system where $M_1=\infty$, and the two-body reduced masses 12 and 13 are $M$ and $3M$ ($M=$nucleon mass), respectively. The two-body subsystems 12 and 13 have a virtual $s$-state and a $p$-resonance, respectively, at the energies indicated by the crosses in the figure. The thick solid circle shows the sum of the two two-body energies. When an attractive interaction between particles 2 and 3 is included the $1^-$ three-body resonances are indicated by the solid circle curve when four terms are included in the expansion (4), and the open circle curve when only one term in (4) is included.

In principle, as in the previous case, one could expect the three-body state at an energy $E$ to be a three-body resonance (it should be a $1^-$ resonance). However the calculation shows that this state is certainly not corresponding to such a resonance. Actually, when the interaction $V_{23}$ is put equal to zero no three-body resonances are found. If we now put some attractive potential between particles 2 and 3 it is then possible to find a $1^-$ three-body state. When the $V_{23}$ interaction is made weaker and weaker we observe how the three-body state is actually moving towards the two-body $p$-resonance instead of towards the three-body
state indicated by the thick solid circle in Fig. 2. This fact is proving that such a point is not a three-body resonance, and can be interpreted as the two-body virtual s-state observed from the two-body p-state (direct calculation of virtual states with the complex scaling method is not possible, since the required rotation angle, larger than $\pi/2$, creates important numerical problems). The results shown by the solid circles in Fig. 2 are obtained by including four terms in the expansion in Eq.(4). When only one term is included the results given by the open circles are obtained. As seen in the figure, the more the two-body energy is approached the more relevant the effect of the higher terms in the expansion (4).

When the mass $M_1$ is made finite but keeping constant the 12 and 13 reduced masses the contribution from the motion of the two-body center of masses is again modifying the three-body energy. However, for all the cases computed we have obtained the same behaviour as for the $M_1=\infty$ case, i.e., when $V_{23}=0$ no resonances are found, and an attractive two-body interaction $V_{23}$ is needed to obtain a $1^-$ three-body resonance. Of course this is not implying that it is not possible to find a case in which the $1^-$ resonance can show up as a result of the center of mass motion.

5 Additional calculations

The same kind of calculations can be performed for some more particular cases, like for instance when 2 and 3 are two identical particles with finite spin. Provided that the two-body interactions are spin-independent the results are similar to the ones shown in Figs. 1 and 2, since the spin part of the wave function is decoupled from the coordinate part and can be used to establish the proper symmetry or antisymmetry between particles 2 and 3. These systems can be used as starting points to construct more realistic systems, like $^6$He or $^{11}$Li, and see how their corresponding three-body states appear when the main properties of the two-body subsystems are well reproduced.

References